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THE RECOVERY-BASED DISCONTINUOUS GALERKIN METHOD FOR THE NAVIER–STOKES EQUATIONS

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The Discontinuous Galerkin (DG) method was first developed by Reed and Hill [1] for advection-type equations, for which it is pre-eminently suited. It combines the best of both finite-volume (FV) and finite-element (FE) methods to be highly accurate, yet compact, flexible and versatile. DG was soon applied to solving diffusion problems, as it is desirable to discretize advection-diffusion equations like the Navier–Stokes equations by one numerical strategy. However, the discontinuous solution representation by DG, its very claim to fame and widespread usage, prevents a straightforward, stable discretization. To overcome this problem, a special step is required that derives common quantities at the interface.

The Recovery-based Discontinuous Galerkin (RDG) method was invented by Van Leer in 2005 [2], and proved to be stable [3]. The required unique interface quantities are "recovered" by a smooth interpolating function f, which is in the *weak* sense identical to the original discontinuous solutions in the pair of elements spanning the interface. A typical example in one dimension for a piece-wise linear solution is shown in Figure (1); for neighbors Ω_1 and Ω_2, f is the solution of the following recovery equations:

$$\int_{\Omega_j} (v_i)_j \, f \, dx = \int_{\Omega_j} (v_i)_j \, u_j \, dx, \;\; i = (1,2) \text{ and } j = (1,2),$$

where $(v_i)_1$ and $(v_i)_2$ are the solution basis functions defined on Ω_1 and Ω_2, respectively. This is a novel idea that breaks the traditional mold of bridging the interface discontinuity; all previous approaches were either rewriting the second-order differential operator as a system of first-order operators (BR2 [4], LDG [4], CDG [5]), or penalizing the discontinuity via an *interior penalty* (IP) method [4].

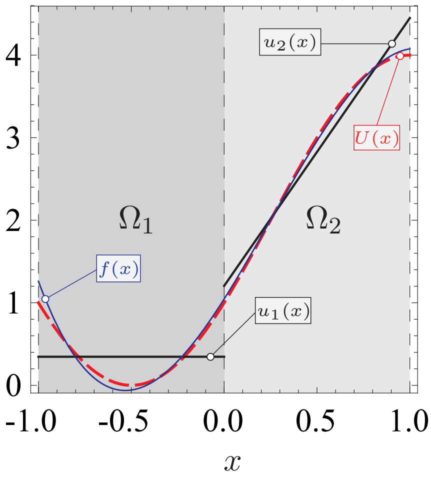

Figure 1: Recovery principle (1D): a sine function U is projected onto two piece-wise *linear* polynomials u_1 and u_2. The resemblance between f and U, especially at the interface, is remarkable, indicating great accuracy potential.

Utilizing a p^{th}-order elemental *tensor-product* polynomial basis, RDG has been demonstrated to achieve the order of accuracy $3p + 2$ or $3p + 1$ for p even or odd, respectively[3], on a Cartesian grid. This is the highest order among all contemporary DG methods for diffusion[6]; Huynh's study also validates the superior stability of RDG. The result is robust: it holds in any number of dimensions[7], for linear as well as nonlinear equations, with or without mixed derivatives. On a triangular grid, however, one has to abandon the tensor-product basis for a *lean* basis of order p to reach an accuracy of the order $2p + 2$ for even and $2p$ for odd p[8]; the latter value may still be improvable. It must be noted that all aforementioned values of the order-of-accuracy are for the cell's average.

In the full paper, we will review RDG and present results of our latest efforts: *i*) to extend RDG to nonlinear diffusion-shear systems on triangular grids, without loss of accuracy; *ii*) to perform direct numerical simulation of compressible turbulence with DG, using recovery for diffusion. To realize the latter in full, the advection discretization must be improved; the standard DG method for advection achieves accuracy of the relatively low order $2p + 1$. One option is to use the $P_N P_M$ method, pioneered by Dumbser[9]; an alternative is to reuse the recovered function for higher accuracy, in combination with appropriate upwinding for stability. We will demonstrate the accuracy, efficiency and robustness of our approach through various test problems (e.g., Taylor-Green vortex, decaying compressible isotropic turbulence).

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