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Principles of Least Action and of Least Constraint - An Excursion into the History of Mechanics

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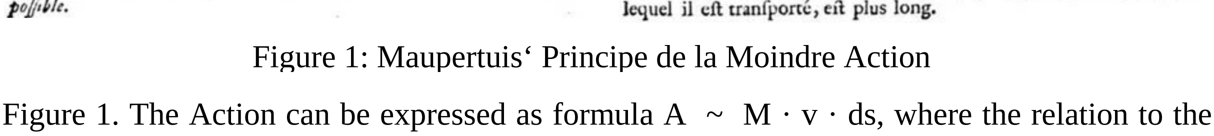
1. SUBJECT OF CONTRIBUTION

The search for the existence of extremum principles in nature and technology goes back to the ancient times. It was not driven exclusively by scientific arguments; also metaphysical considerations played an important role. With the upcoming modern science the issue became a central subject in mathematics and mechanics. Here we refer to two principles suggested in the middle of the 18th and at the beginning of the 19th century. The first one is the Principle of Least Action attributed to Pierre-Louis Moreau de Maupertuis (1698-1759). It resulted in one of the most discussed controversies in the history of mechanics, initiated at the Royal Prussian Academy of Sciences in 1746. The second one, the Principle of Least Constraint, has been proposed 1829 by Carl Friedrich Gauss (1777-1855) in a short article in the "Journal für die reine and angewandte Mathematik". The present contribution is a summary of a paper published recently by the author in [1] in which also key references are listed.

2. PRINCIPLE OF LEAST ACTION

The French mathematician P.-L. Moreau de Maupertuis published papers to the French Academy of Sciences in 1741 and 1744 mentioning a principle minimizing a quantity which he called action. On invitation of King Frederick II ('the Great') he became President of the Prussian Academy of Sciences in Berlin in 1746; shortly thereafter he presented a book *"Les Loix du Movement et du Repos"* (The Laws of Movement and of Rest). In the introduction he refers to Euler's work "*Methodus inveniendi lineas curvas"* published in 1744 dealing with the same subject, however describing minimum and maximum properties. Maupertuis states his "*Principé de la Moindre Action"* as 'When a change occurs in Nature, the Quantity of Action necessary for that change is as small as possible'. He defines 'The Quantity of Action is a product of the Mass of Bodies times their velocity and the distance they travel….'; see


PRINCIPE GENERAL.

Lors qu'il arrive quelque changement dans la Nature, la Quantité d'Action, néceſſaire pour ce changement, eſt la plus petite qu'il ſoit poſſible.

La *Quantité d'Action* eſt le produit de la Maſſe des Corps, par leur vîteſſe & par l'eſpace qu'ils parcourent. Lors qu'un Corps eſt tranſporté d'un lieu dans un autre, l'Action eſt d'autant plus grande, que la Maſſe eſt plus groſſe; que la vîteſſe eſt plus rapide; que l'eſpace, par lequel il eſt tranſporté, eſt plus long.

Figure 1: Maupertuis' Principe de la Moindre Action

Figure 1. The Action can be expressed as formula $A \sim M \cdot v \cdot ds$, where the relation to the kinetic energy is apparent (vis viva, the living force).

Another member of the Academy, the Swiss mathematician Johann Samuel König (1712-

1757), published an essay in Nova Acta Eruditorum in 1751, where he argued that the principle was not correct because it should refer to a minimum and maximum property; and most importantly he claimed that it had already been mentioned 1708 in a letter by Leibniz (1646-1716) to the Swiss mathematician Jacob Hermann (1678-1733). Since König could only provide a copy of the letter he was accused of forgery by a committee of the Academy headed by Euler. On the other hand König was strongly supported by the French writer Voltaire who published several ugly polemics like the "Diatribe", see also the "Maupertuisiana" compiling several defamatory articles. The original letter of Leibniz was never found; its authenticity though was a matter of discussion up to the present time. Thus the case was never clarified.

The Principle of Least Action was not followed in the mechanics community; here mostly d'Alembert's and Hamilton's principles have been used. However it is still mentioned in modern textbooks and applied for example in the theory of relativity and quantum mechanics.

3. PRINCIPLE OF LEAST CONSTRAINT

Carl Friedrich Gauss developed his famous Method of Least Squares in 1795; however it was published not earlier than 1809. Twenty years later he proposed the Principle of Least Constraint as a mechanical analoque to the Least Square Principle. It says *'The motion of a system of material points… takes place in every moment in maximum accordance with the free movement or under least constraint'*. He continues *'the measure of constraint, …., is considered as the sum of products of mass and the square of the deviation to the free motion'*. Applying the usual mathematical notation the acceleration a_i^{free} of the free unconstrained motion is defined by the F_i divided by the mass m_i. The motion with a_i is constraint by extra kinematical conditions, e.g. prescribed displacements. The mass m_i in the sum serves as weight. Z is the constraint (called "Zwang" in the original paper) which is supposed to be a

$$Z \sim \sum_{i=1}^{N} \left(a_i - a_i^{\text{free}}\right)^2 \qquad \text{or} \qquad Z = \sum_{i=1}^{N} m_i \left(a_i - \frac{F_i}{m_i}\right)^2 = \text{MIN}$$

minimum. The analogy to the Least Square Principle is obvious. Gauss refers to this observation in his paper saying *'It is strange that the free movements, when they cannot withstand the necessary conditions, are modified in the same way as the analyzing mathematician, applying the method of least squares, balances experiences which are based on parameters depending on necessary interactions'*. He also says *'This analogy could be further followed up, but this is currently not my intention'*. He never picked it up again.

It can be shown that there is a strong relation of the Principle of Least Constraint to d'Alembert's Principle [1]. This might be the reason that the Principle has not been applied too often, although it is mentioned as fundamental principle in many treatises up to present time. However even today it is applied in physics, e.g. in statistical mechanics.

REFERENCE

[1] E. Ramm, Principles of Least Action and of Least Constraint. In *The History of Theoretical, Material and Computational Mechanics – Mathematics meets Mechanics and Engineering*. Erwin Stein (ed.), Lecture Notes in Applied Mathematics and Mechanics (LAMM),Vol. 1, Springer 2014, pp 23-43.