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ON THE COMPUTATION OF STABILITY POINTS WITH LEAST-SQUARES MIXED FINITE ELEMENT METHODS

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In the present contribution least-squares mixed finite element formulations are investigated with respect to stability issues in hyperelastic problems. A main point will be the computation of so-called stability points, which are indicated by critical load stages where the eigenvalues decrease drastically. Besides that the ensurance of the stability of the numerical scheme (avoiding of stress oscillations) will be investigated.
In the recent years the least-squares method has been applied to many problem fields in mechanics and mathematics. Exhaustive overviews with respect to theoretical foundations and application-oriented issues can be found for example in the monograph [1]. The method provides some (theoretical) advantages compared to other variational approaches, e.g. there is no restriction to the LBB-condition and an a posteriori error estimator is provided without any additional computational costs. Basis for the method is the minimization of a so-called least-squares functional consisting of the equilibrium condition and the constitutive relation for a hyperelastic material, e.g.

$$\mathcal{F}(\boldsymbol{P},\boldsymbol{u}) \;=\; \frac{1}{2}\left(\int_{\mathcal{B}} |w_1(\operatorname{Div}\boldsymbol{P}+\boldsymbol{f})|^2 dv + \int_{\mathcal{B}} |w_2(\boldsymbol{P}-\rho_0\partial_F\psi(\boldsymbol{C}))|^2 dv\right), \qquad (1)$$

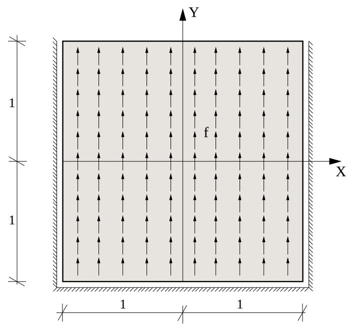
with the first Piola-Kirchhoff stress tensor $\boldsymbol{P}$, the body force $\boldsymbol{f}$, the deformation gradient $\boldsymbol{F}$, the right Cauchy-Green tensor $\boldsymbol{C}=\boldsymbol{F}^T\boldsymbol{F}$, the positive weights w_i and the free energy ψ. The choice of suitable weights has been considered e.g. in [2]. The minimization of (1) requires the first variation with respect to the displacements and the stresses to be zero. Supported by our previous experience, the most natural combination of finite elements is to use Raviart-Thomas elements for the stresses and standard continuous elements for the displacements. This leads to conforming approximations

$$(\boldsymbol{P}_h,\boldsymbol{u}_h)\in\boldsymbol{X}_h^m\times\boldsymbol{V}_h^k\subset W^p(\operatorname{Div},\mathcal{B})\times W^{1,p}(\mathcal{B}) \quad \text{with} \quad p\geq 4, \qquad (2)$$

denoted in the sequel by $RT_m P_k$, with respect to our solution spaces.
The contribution covers several numerical examples, as for instance a model problem, see e.g. [3] and [4], for finite strain incompressible elasticity. This bidimensional example is able to show, despite of its simplicity, some of the problems appearing for incompressible materials in case of nonlinear elasticity. Here, the results show the stability points in the analysis and the non-physical eigenfunctions for the critical load values.

Boundary conditions, material parameters and system:

- Left / right side: $u_1 = 0$, $\boldsymbol{P}\,\boldsymbol{N} \cdot \boldsymbol{t} = 0$
- Lower side: $u_2 = 0$, $\boldsymbol{P}\,\boldsymbol{N} \cdot \boldsymbol{t} = 0$
- Upper side: $\boldsymbol{P}\,\boldsymbol{N} = (0, 0)$
- Body force: $\boldsymbol{f} = (0, 8)$
- $\lambda = 10^8$, $\mu = 1$



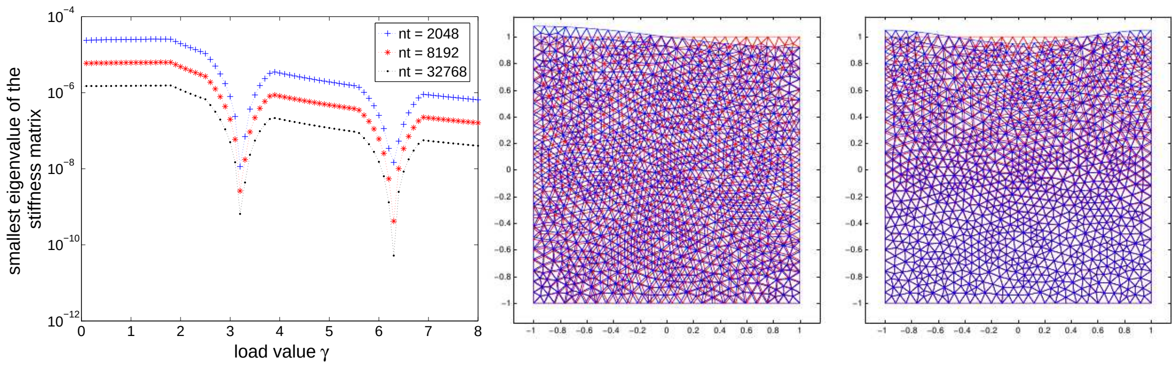


Figure 1: Setup of problem (upper right), stability points (lower left), eigenfunctions (lower right).

REFERENCES

[1] P.B. Bochev and M.D. Gunzburger, Least-Squares Finite Element Methods, *Springer Verlag*, New York (2009).

[2] K. Steeger, A. Schwarz, J. Schröder, G. Starke and B. Müller, Solving hyperelastic problems using mixed LSFEM, *Proc. Appl. Math. Mech.* Vol. 12 pp. 215-216, (2012).

[3] F. Auricchio, L. Beirão da Veiga, C. Lovadina and A. Reali, The Importance of the Exact Satisfaction of the Incompressibility Constraint in Nonlinear Elasticity: Mixed FEMs versus NURBS-Based Approximations *Comput. Meth. Appl. Mech. Engrg.* Vol. 199 pp. 314-323, (2010).

[4] B. Müller, G. Starke, A. Schwarz and J. Schröder, A first-order system least squares method for hyperelasticity, *submitted*, (2013).