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ACCURATE EXPLICIT FINITE ELEMENT METHOD FOR WAVE PROPAGATION AND DYNAMIC CONTACT PROBLEMS

RADEK KOLMAN*, SANG SOON CHO† AND K.C. PARK‡

*Institute of Thermomechanics
Academy of Sciences of the Czech Republic, v.v.i.
Dolejskova 5, Prague 8, Czech Republic
e-mail: kolman@it.cas.cz, web page: http://www.it.cas.cz/en/kolmanr

†Korea Atomic Energy Research Institute
999-111 Daedeok-Daero, Yuseong-gu, Daejeon, Republic of Korea
e-mail: sscho96@kaeri.re.kr

‡Center for Aerospace Structures
University of Colorado at Boulder
Boulder, CO 80309 - 0429, Colorado, USA
e-mail: kcpark@colorado.edu - web page:
http://www.colorado.edu/engineering/CAS/KCPark.d/KCParkHome.d/KCPark.html

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Abstract. An accurate explicit integration algorithm in the predictor-corrector form for finite element computations of wave propagation and contact problems in solids is presented. The nominated algorithm, with the component-wise partition of equations of motion to longitudinal and shear parts, is designed to more precisely integrate wave propagation in accordance with their different propagation wave speeds and their stability limits. The suggested three-time step integrator is fully explicit with a diagonal mass matrix requirement, second-order accurate, conditionally stable and exhibits minimal sensitivity behaviour on the time step size satisfying the stability limit. The submitted time algorithm is able to be easily implemented into standard finite element codes for general non-linear dynamics problems - wave propagation, dynamic plasticity with small and large deformations, dynamic crack propagation with cohesive fracture models and impact/contact problems. In a numerical test of wave propagation in a disc, we compare results obtained by the proposed scheme with existing conventional time integrators.

1 INTRODUCTION

The direct time integration is an up-to-date standard tool in finite element (FE) solution of transient and wave propagation problems [1]. An explicit method is implemented in an each FE software (e.g. ABAQUS, ANSYS, LS-DYNA, PAM-CRASH, TAHOE, ...). The contribution deals with an explicit algorithm suitable to suppress spurious oscillations in FE computations of linear and non-linear wave propagation problems in solids. The eliminating and diminishing of spurious oscillations in FE simulations of discontinuous wave propagation is an up-to-date open problem to research study.

Frequently, time integration schemes are categorized into explicit methods [2], implicit methods [4], and implicit-explicit methods. Essentially, the implicit method needs factorization of an effective matrix, one or more times per step, and the method is explicit in the opposite case, employed a diagonal (lumped) mass matrix and a vector solver [1]. As for the time integration of discretized equations of motion, for wave propagation in solids, the central difference (CD) method [2] is most widely and commonly used for explicit treatment in time and the Newmark (average acceleration) method [3, 4] for implicit time integration. Implicit computation needs a much larger computation effort per time step due to the solving a linear matrix system. However, implicit methods are derived to have unconditional stability then the time step size is chosen freely based only on the physical problem to be solved [1]. On the other hand, explicit methods need only a vector solver, but mostly are merely conditional stable. Thus, the time step size has to satisfy a stability limit [2].

Among explicit time integration methods, the nearly universe choice is the central difference method which possesses no numerical dissipation for linear FE discretization with a diagonal mass matrix. In one-dimensional case, if a linear finite element mesh is generated so that the stability limit is the same for each element, results obtained by the central difference with the diagonal mass matrix are very close to exact solutions. The optimal time-consuming and natural choice of the time step size is then equal to the stability limit. For that case, spatial dispersion errors of the linear FEM with the diagonal mass matrix [5] and period elongation behavior of the central difference method [1] suppress each other, but only in one-dimensional case [6]. However, in practice, it is not feasible to employ a mesh so that the critical time step limit is the same for all elements, thus results by the central difference method produce spurious oscillations.

By assuming a theoretical prediction in multidimensional wave propagation in unbounded continuum, longitudinal and shear waves propagate with different speeds, c_L and c_S [7]. Therefore, the mismatch in the wave speeds of the two types of wave components produces spurious oscillations in numerical results. In the central difference method, a single-time step computation with a time step size, which is given by the fastest propagating longitudinal wave, is employed. Thus longitudinal waves are integrated more accurate then shear ones. This is the main disadvantage of the central difference method using in modelling of multidimensional wave propagation problems in solids. For that reason, a modification of the central difference method based on the component-wise partition of equations of motion to longitudinal and shear parts has been found [8, 9]. In this paper, we present the predictor-corrector form of the nominated explicit time scheme exploited quadrilateral and hexahedral finite elements. Moreover, results obtained by the time scheme inhibit excellent stress and strain histories with minimum spurious oscillations near theoretical wavefronts.

2 FINITE ELEMENT METHOD IN SOLID DYNAMICS

Spatial discretization of a general dynamic problem by FEM introduces the second-order ordinary differential system [10]

$$\begin{aligned} \mathbf{M}\ddot{\mathbf{u}}^n = \mathbf{f}_{ext}(t^n) - \mathbf{f}_{int}\left(\mathbf{u}^n, \dot{\mathbf{u}}^n, t^n\right), \quad t^n \in [t^0, T] \\ \mathbf{u}(t^0) = \mathbf{u}^0,\ \dot{\mathbf{u}}(t^0) = \mathbf{v}^0 \end{aligned} \tag{1}$$

where $\mathbf{M}$ is the mass matrix, $\mathbf{f}_{ext}$ is the time-dependent external load vector, $\mathbf{f}_{int}$ is the vector of generally non-linear internal forces, the vectors $\mathbf{u}^n$, $\dot{\mathbf{u}}^n$ and $\ddot{\mathbf{u}}^n$ contain nodal displacements, velocities and accelerations at the time t^n, respectively. Quantities with the superscript n have a meaning of the approximation of quantities at the time t^n, e.g. $\mathbf{u}^n \approx \mathbf{u}(t^n)$ and so on. Vectors $\mathbf{u}^0$ and $\mathbf{v}^0$ get values of the initial state for nodal displacements and velocities at the time t^0. The discretized time is considered as $t^n = n\Delta t$, where Δt is the time step size. The external loading $\mathbf{f}_{ext}$ is usually a consequence of application of traction boundary conditions, body forces or contact forces. We assume a diagonal mass matrix $\mathbf{M}$.

3 THE CENTRAL DIFFERENCE METHOD

In this paper, the main attention is paid to explicit FE computations, therefore we introduce the algorithm of the central difference method. The predictor-corrector form of the central difference method [1] to solve a general dynamic problem with geometrical and material non-linearities is following.

Predictor phase:

$$\begin{aligned} \tilde{\mathbf{u}}_{cd}^{n+1} &= \mathbf{u}^n + \Delta t\dot{\mathbf{u}}^n + \frac{\Delta t^2}{2}\ddot{\mathbf{u}}^n \\ \dot{\tilde{\mathbf{u}}}_{cd}^{n+1} &= \dot{\mathbf{u}}^n + \frac{\Delta t}{2}\ddot{\mathbf{u}}^n \\ \ddot{\tilde{\mathbf{u}}}_{cd}^{n+1} &= \mathbf{0} \end{aligned} \tag{2}$$

The system of equations of motion constituted at the time t^{n+1} to solve:

$$\mathbf{M}\Delta\ddot{\tilde{\mathbf{u}}}_{cd}^{n+1} = \mathbf{f}_{ext}^{n+1} - \mathbf{f}_{int}(\tilde{\mathbf{u}}_{cd}^{n+1}, \dot{\tilde{\mathbf{u}}}_{cd}^{n+1}, t^{n+1}) \tag{3}$$

Corrector phase:

$$\begin{aligned} \mathbf{u}_{cd}^{n+1} &= \tilde{\mathbf{u}}_{cd}^{n+1} \\ \dot{\mathbf{u}}_{cd}^{n+1} &= \dot{\tilde{\mathbf{u}}}_{cd}^{n+1} + \frac{\Delta t}{2}\Delta\ddot{\tilde{\mathbf{u}}}_{cd}^{n+1} \\ \ddot{\mathbf{u}}_{cd}^{n+1} &= \Delta\ddot{\tilde{\mathbf{u}}}_{cd}^{n+1} \end{aligned} \tag{4}$$

The aforementioned process is fully explicit with a diagonal mass matrix. In addition, the combination of the central difference method and the diagonal mass matrix of linear FEM produces a providential numerical effect due to the spatial and temporal discretization. In this case, dispersion errors are mutually repressed by a suitable choice of the time step size [6].

Generally, explicit methods are only conditionally stable; the time step size Δt has to satisfy a stability limit in the form $\Delta t \leq \Delta t_c$, where Δt_c is the critical time step size. The critical time step size Δt_c securing the stability of the central difference method for a linear undamped system takes the form [11]

$$\Delta t_c = \frac{2}{\omega_{max}} \tag{5}$$

where ω_{max} being the maximum eigenfrequency of the system, related to the generalized eigenvalue problem $\mathbf{K}\mathbf{u} = \lambda \mathbf{M}\mathbf{u}$, taking $\omega^2 = \lambda$ and the stiffness matrix $\mathbf{K}$ of the linearized system (1) [1].

In wave propagation problem in solids, the stability limit is approximately equal to a time required to run longitudinal elastic wave with the speed c_L through the smallest finite element constituting a FE mesh [1]. For some element types and uniform FE meshes, the critical time step size is estimated under elastic wave propagation rule of thumb $\Delta t_c = H/c_L$, where H is the characteristic length of the smallest element of a FE mesh. For uniform 4-nodes square quadrilateral and 8-nodes cubic hexahedral finite elements, the stability limit is estimated as $\Delta t_c = H/c_L$, where H is the element edge [1].

4 A NOVEL TIME INTEGRATION SCHEME

In this section, the predictor-corrector form of the time scheme suggested in [9] is mentioned. Moreover, we describe it in a step-by-step flowchart (Tab. 1), how we have to implement the algorithm into standard finite element codes exploited quadrilateral and hexahedron linear finite elements and a diagonal mass matrix.

In [8, 9], it has been brought about an idea a modification of the conventional central difference method being able to suppress spurious oscillations in numerical solution by lower-order finite elements. This mentioned time integration is based on component-wise partitioned equations of motion (1) to longitudinal and shear component of an approximated displacement field and also based on pullback time integrations [8]. The elemental component-wise partitioned equations set have to form [9, 12]

$$\begin{aligned} \text{Longitudinal component equation:} \quad & \mathbf{M}^e \ddot{\mathbf{u}}^e_L = \mathbf{f}^e_{ext,L} - \mathbf{f}^e_{int,L} \\ \text{Shear component equation:} \quad & \mathbf{M}^e \ddot{\mathbf{u}}^e_S = \mathbf{f}^e_{ext,S} - \mathbf{f}^e_{int,S} \end{aligned} \tag{6}$$

where subscripts L and S correspond to longitudinal and shear approximated displacement fields, $\mathbf{f}^e_L$ and $\mathbf{f}^e_S$ mark decomposed forces, so that $\mathbf{f}^e = \mathbf{f}^e_L + \mathbf{f}^e_S$, and $\mathbf{M}^e$ denotes the elemental mass matrix. Practically, force decomposition is given by multiplying elemental force vectors by decomposed matrices for a longitudinal and shear part, $\mathbf{D}^e_L$ and $\mathbf{D}^e_S$, as [9, 12]

$$\begin{aligned} \mathbf{f}^e_{ext,L} = \mathbf{D}^{e\mathrm{T}}_L \mathbf{f}^e_{ext}, \quad & \mathbf{f}^e_{int,L} = \mathbf{D}^{e\mathrm{T}}_L \mathbf{f}^e_{int} \\ \mathbf{f}^e_{ext,S} = \mathbf{D}^{e\mathrm{T}}_S \mathbf{f}^e_{ext}, \quad & \mathbf{f}^e_{int,S} = \mathbf{D}^{e\mathrm{T}}_S \mathbf{f}^e_{int} \end{aligned} \tag{7}$$

Then global vectors of external and internal decomposed forces are assembled in a standard FE fashion to the vectors $\mathbf{f}_{ext}$ and $\mathbf{f}_{int}$.

Longitudinal and shear component-wise partitioning matrices, $\mathbf{D}_L^e$ and $\mathbf{D}_S^e$, possess the following properties [9]:

$$\text{Partition of unity:} \quad \mathbf{D}_S^e + \mathbf{D}_L^e = \mathbf{I}^e \tag{8}$$
$$\text{Projector property:} \quad \mathbf{D}_S^{e\mathrm{T}}\mathbf{D}_S^e = \mathbf{D}_S^e,\ \mathbf{D}_L^{e\mathrm{T}}\mathbf{D}_L^e = \mathbf{D}_L^e \tag{9}$$
$$\text{Symmetry:} \quad \mathbf{D}_S^{e\mathrm{T}} = \mathbf{D}_S^e,\ \mathbf{D}_L^{e\mathrm{T}} = \mathbf{D}_L^e \tag{10}$$
$$\text{Orthogonality:} \quad \mathbf{D}_L^e\mathbf{D}_S^e = \mathbf{D}_S^e\mathbf{D}_L^e = \mathbf{0}^e \tag{11}$$
$$\text{Element mass commutability:} \quad \mathbf{D}_L^{e\mathrm{T}}\mathbf{M}^e = \mathbf{M}^e\mathbf{D}_L^e,\ \mathbf{D}_S^{e\mathrm{T}}\mathbf{M}^e = \mathbf{M}^e\mathbf{D}_S^e \tag{12}$$
$$\text{Element mass orthogonality:} \quad \mathbf{D}_L^{e\mathrm{T}}\mathbf{M}^e\mathbf{D}_S^e = \mathbf{M}^e\mathbf{D}_L^e\mathbf{D}_S^e = \mathbf{0}^e \tag{13}$$

where $\mathbf{I}^e$ is the unity matrix.

In Table 1, the nominated three-time step algorithm is described in details and its step-by-step flowchart is depicted. The time scheme is applicable in a fully predictor-corrector form for a general non-linear dynamic problem. The variable time step size Δt^n is assumed to be adopted at each time step.

Table 1: A step-by-step flowchart of the suggested scheme in the predictor-corrector form.

A. *Initial calculation at the time t^0*

1. Initialize $\mathbf{u}^0$, $\dot{\mathbf{u}}^0$, all internal variables from the restart and compute $\mathbf{M}^0$.
2. Compute $\mathbf{f}^{int}(\mathbf{u}^0, \dot{\mathbf{u}}^0, t^0)$, $\mathbf{f}^{ext}(t^0)$, and $\ddot{\mathbf{u}}^0 = (\mathbf{M}^0)^{-1}(\mathbf{f}^{ext} - \mathbf{f}^{int})$.
3. Decompose $\ddot{\mathbf{u}}^0$ so that $\ddot{\mathbf{u}}^0 = \ddot{\mathbf{u}}_L^0 + \ddot{\mathbf{u}}_S^0$

B. *For each time step*

⟨**Set required time step sizes**⟩

1. Estimate stability limits as $\Delta t_L^n = H^n/c_L^n$ and $\Delta t_S^n = H^n/c_S^n$.
2. Set time step size $\Delta t^n = \alpha_L^n \Delta t_L^n$ so that $\alpha_L^n \in (0, 1]$.
3. Compute $\alpha_L^n = \Delta t^n/\Delta t_L^n$ and $\alpha_S^n = \Delta t^n/\Delta t_S^n$.

⟨**First sub-step**⟩ - Front-shock including integration of longitudinal component

1. We know $\mathbf{u}^n$, $\dot{\mathbf{u}}^n$, $\ddot{\mathbf{u}}^n$, $\ddot{\mathbf{u}}_L^n$, and all internal variables at the time t^n.
2. Set $t^{n+L} = t^n + \Delta t_L^n, \quad \mathbf{u}_{fs}^{n+1} = \mathbf{u}^n + \Delta t^n \dot{\mathbf{u}}^n$.
3. Predictor phase:

$$\tilde{\mathbf{u}}^{n+L} = \mathbf{u}^n + \Delta t_L^n \dot{\mathbf{u}}^n + \tfrac{1}{2}\left(\Delta t_L^n\right)^2 \ddot{\mathbf{u}}^n$$
$$\tilde{\dot{\mathbf{u}}}^{n+L} = \dot{\mathbf{u}}^n + \tfrac{1}{2}\Delta t_L^n \ddot{\mathbf{u}}^n$$
$$\tilde{\ddot{\mathbf{u}}}^{n+L} = \mathbf{0}$$

Apply boundary conditions at the time t^{n+L} and update coordinates $\mathbf{x}^{n+L} = \mathbf{X} + \tilde{\mathbf{u}}^{n+L}$.

Continuity on the next page

Table 1. Continuity of the step-by-step flowchart of the suggested scheme in the predictor-corrector form.

4. Solve the update vector:
   $\mathbf{M}^n \Delta \ddot{\tilde{\mathbf{u}}}_L^{n+L} = \mathbf{f}_L^{ext}(t^{n+L}) - \mathbf{f}_L^{int}(\tilde{\mathbf{u}}^{n+L}, \dot{\tilde{\mathbf{u}}}^{n+L}, t^{n+L})$
5. Corrector phase:
   $\ddot{\mathbf{u}}_L^{n+L} = \Delta \ddot{\tilde{\mathbf{u}}}_L^{n+L}$
   Apply boundary conditions at the time t^{n+L}.
   Pullback approximation:
   $\mathbf{u}_{fs}^{n+1} = \mathbf{u}_{fs}^{n+1} + (\Delta t_L^n)^2 \beta_1(\alpha_L^n) \ddot{\mathbf{u}}_L^n + (\Delta t_L^n)^2 \beta_2(\alpha_L^n) \ddot{\mathbf{u}}_L^{n+L}$
   $\beta_1(\alpha_L^n) = \frac{1}{6}\alpha_L^n\big(1 + 3\alpha_L^n - (\alpha_L^n)^2\big), \beta_2(\alpha_L^n) = \frac{1}{6}\alpha_L^n\big((\alpha_L^n)^2 - 1\big)$
6. Save $\mathbf{u}_{fs}^{n+1}$.
7. Reset all variables to the state at the time t^n. (Do not update internal variables).

⟨**Second sub-step**⟩ - Front-shock including integration of shear component

1. We know $\mathbf{u}^n$, $\dot{\mathbf{u}}^n$, $\ddot{\mathbf{u}}^n$, $\ddot{\mathbf{u}}_S^n$, and all internal variables at the time t^n.
2. Set $t^{n+S} = t^n + \Delta t_S^n$.
3. Predictor phase:
   $\tilde{\mathbf{u}}^{n+S} = \mathbf{u}^n + \Delta t_S^n \dot{\mathbf{u}}^n + \frac{1}{2}\left(\Delta t_S^n\right)^2 \ddot{\mathbf{u}}^n$
   $\dot{\tilde{\mathbf{u}}}^{n+S} = \dot{\mathbf{u}}^n + \frac{1}{2}\Delta t_S^n \ddot{\mathbf{u}}^n$
   $\ddot{\tilde{\mathbf{u}}}^{n+S} = \mathbf{0}$
   Apply boundary conditions at the time t^{n+S} and update coordinates $\mathbf{x}^{n+S} = \mathbf{X} + \tilde{\mathbf{u}}^{n+S}$.
4. Solve the update vector:
   $\mathbf{M}^n \Delta \ddot{\tilde{\mathbf{u}}}_S^{n+S} = \mathbf{f}_S^{ext}(t^{n+S}) - \mathbf{f}_S^{int}(\tilde{\mathbf{u}}^{n+S}, \dot{\tilde{\mathbf{u}}}^{n+S}, t^{n+S})$
5. Corrector phase:
   $\ddot{\mathbf{u}}_S^{n+S} = \Delta \ddot{\tilde{\mathbf{u}}}_S^{n+S}$
   Apply boundary conditions at the time t^{n+L}.
   Pullback approximation:
   $\mathbf{u}_{fs}^{n+1} = \mathbf{u}_{fs}^{n+1} + (\Delta t_S^n)^2 \beta_1(\alpha_S^n) \ddot{\mathbf{u}}_S^n + (\Delta t_S^n)^2 \beta_2(\alpha_S^n) \ddot{\mathbf{u}}_S^{n+S}$
   $\beta_1(\alpha_S^n) = \frac{1}{6}\alpha_S^n\big(1 + 3\alpha_S^n - (\alpha_S^n)^2\big), \beta_2(\alpha_S^n) = \frac{1}{6}\alpha_S^n\big((\alpha_S^n)^2 - 1\big)$
6. Save $\mathbf{u}_{fs}^{n+1}$.
7. Reset all variables to the state at the time t^n. (Do not update internal variables).

⟨**Third sub-step**⟩ - Post-shock including integration with the averaging for given $\theta \in [0, 1]$

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Table 1. Continuity of the step-by-step flowchart of the suggested scheme in the predictor-corrector form.

1. We know $\mathbf{u}^n$, $\dot{\mathbf{u}}^n$, $\ddot{\mathbf{u}}^n$, $\mathbf{u}_{fs}^{n+1}$, and all internal variables at the time t^n.
2. Set $t^{n+1} = t^n + \Delta t_1^n$.
3. Predictor phase:
   $\tilde{\mathbf{u}}_{cd}^{n+1} = \mathbf{u}^n + \Delta t^n \dot{\mathbf{u}}^n + \frac{1}{2}\left(\Delta t^n\right)^2 \ddot{\mathbf{u}}^n$
   $\tilde{\mathbf{u}}^{n+1} = \theta \mathbf{u}_{fs}^{n+1} + (1-\theta)\tilde{\mathbf{u}}_{cd}^{n+1}$
   $\dot{\tilde{\mathbf{u}}}^{n+1} = \dot{\mathbf{u}}^n + \frac{1}{2}\Delta t^n \ddot{\mathbf{u}}^n$
   $\ddot{\tilde{\mathbf{u}}}^{n+1} = \mathbf{0}$
   Apply boundary conditions at the time t^{n+1} and update coordinates $\mathbf{x}^{n+1} = \mathbf{X} + \tilde{\mathbf{u}}^{n+1}$.
4. Solve the update vector:
   $\mathbf{M}^n \Delta \ddot{\tilde{\mathbf{u}}}^{n+1} = \mathbf{f}^{ext}(t^{n+1}) - \mathbf{f}^{int}(\tilde{\mathbf{u}}^{n+1}, \dot{\tilde{\mathbf{u}}}^{n+1}, t^{n+1})$
5. Corrector phase:
   $\mathbf{u}^{n+1} = \tilde{\mathbf{u}}^{n+1}$
   $\dot{\mathbf{u}}^{n+1} = \dot{\tilde{\mathbf{u}}}^{n+1} + \frac{1}{2}\Delta t^n \Delta \ddot{\tilde{\mathbf{u}}}^{n+1}$
   $\ddot{\mathbf{u}}^{n+1} = \Delta \ddot{\tilde{\mathbf{u}}}^{n+1}$
   Apply boundary conditions at the time t^{n+L}.
   Decompose $\ddot{\mathbf{u}}^{n+1}$ so that $\ddot{\mathbf{u}}^{n+1} = \ddot{\mathbf{u}}_L^{n+1} + \ddot{\mathbf{u}}_S^{n+1}$.
6. Save $\mathbf{u}^{n+1}$, $\dot{\mathbf{u}}^{n+1}$, $\ddot{\mathbf{u}}^{n+1}$, $\ddot{\mathbf{u}}_L^{n+1}$, and $\ddot{\mathbf{u}}_S^{n+1}$. Update internal variables.

As we said sooner, the new suggested algorithm is a three-time step scheme. The first two time sub-steps require the pullback integrations for each partitioned equations of motion. In that cases, longitudinal and shear components are computed with the different time steps given by each stability limit, Δt_L and Δt_S. The last time sub-step computation needs the pushforward integration (the central difference method) with a time step size $\Delta t < \Delta t_L$.

The presented three-time step integrator is completely explicit with a diagonal mass matrix requirement, second-order accurate, conditionally stable and it shows a minimum sensitivity behavior on the time step size. 4-noded quadrilateral and 8-noded hexahedral under-integrated elements are then preferred option to model solid components. In multidimensional tasks, the proposed algorithm utilizes the component-wise partition of equations of motion to the longitudinal and shear part. Moreover, the each component of equations of motion is integrated separately with each stability limit. It means with different time step sizes, due the mitigating dispersion errors and spurious oscillations. The algorithm has been successful implemented into an open research code Tahoe [13]. The detail comments to implementation are mentioned in [12].

5 A NUMERICAL TEST – A THIN ELASTIC DISC LOADED BY A SUDDEN RADIAL FORCE

In the benchmark test of the paper, we test accuracy and performance of the proposed time integrator in elastic wave propagation in a plane domain with curved boundaries and on an unstructured mesh consisted of 4-noded linear elements. A thin elastic disc suddenly loaded by a constant normal stress is considered, for scheme see Figure 1. The task is assumed as plane stress state and the time history of normal stress is taken as the Heaviside step function.

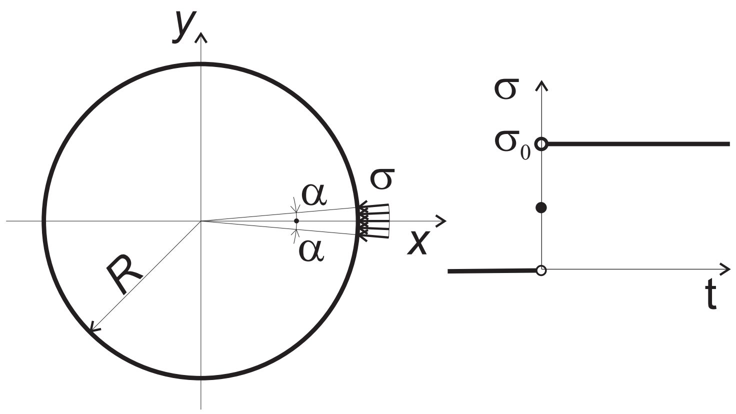
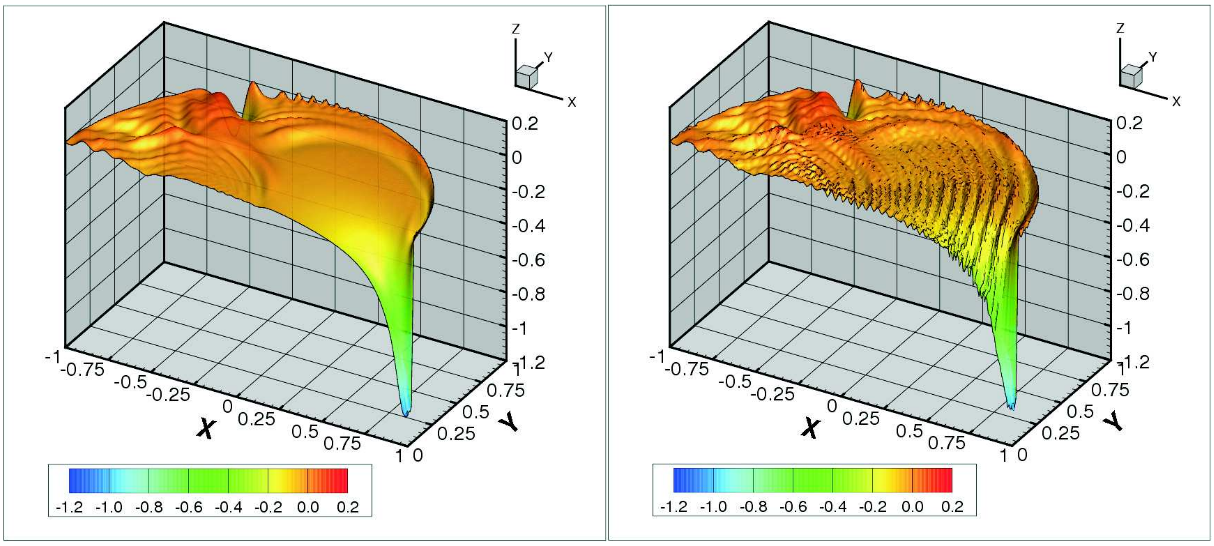


Figure 1: Problem definition of a thin elastic disc loaded by a sudden radial force.

In this linear task, model parameters are chosen as non-physical: the disc radius $R = 1$ mm, the amplitude of normal stress (pressure) $\sigma_0 = 1$ GPa, the angle of the applied normal stress $\alpha = \pi/60$, the Young's modulus $E = 8/9$ GPa, the Poisson's ratio $\nu = 1/3$, and the mass density $\rho = 1$ mg/mm^3. The material parameters were set so that corresponding wave speeds for a plane stress problem taken values $c_L = 1$ mm/μs, $c_S = 1/\sqrt{3} = 0.5774$ mm/μs and the approximated value of Rayleigh's wave speed on a straight boundary was given as $c_R = 0.5369$ mm/μs.

Due to symmetry of the model, only an one half of the disc is discretized by 21600 4-noded linear elements with 21841 nodes. The nodes on the symmetry line (x-axis) are fixed in the perpendicular direction to the line. The boundary of the model on the x-axis is uniformly approximated by 240 elements. Thus, the characteristic length of the mesh is $H = R/120 = 1/120 = 0.00833$ mm. The FE mesh was generated so that distortion of finite elements is the smallest. The time step size is taken with respect to the stability limit (5). An approximate estimate of the stability limit of the time step size for this model, the finite element mesh and the material parameters was set up by few computations by the central difference method as $\Delta t_{cr} = 0.7\, H/c_L$. Thus the time step size for the proposed method is takes as $\Delta t = 0.5 \Delta t_{cd} = 0.35\, H/c_L = 7/2400$ μs. The following estimations of the critical time steps for the proposed time integration are utilized as $\Delta t_L = \Delta t_{cr}$ and $\Delta t_S = \Delta t_L c_L / c_S$, thus $\alpha_L = \Delta t / \Delta t_L = 0.5$ and $\alpha_S = \Delta t / \Delta t_S = 0.288675$.



(a) (b)

Figure 2: Distributions of non-dimensional stress σ_{xx}/σ_0 at the disc at the time $4\,R/c_L$: (a) the proposed method, (b) the central difference method. A half of a disc is shown.

A response of an elastic disc to impact and moving loads has been studied in [14, 15], where the analytical derivation of dynamic stress states and displacement distributions versus time have been published. The numerical evaluation of stress and displacement distributions versus time has been presented by Červ in [16]. For the sake of completeness, a response of a visco-elastic disc under sudden radial loading has been studied in [17] and dispersion of Rayleigh's wave in a disc has been published in [18].

As an example, numerical results of the elastic wave propagation problem in the disc are shown in Figure 2. One can see distributions of non-dimensional stress σ_{xx}/σ_0 at the time $4\,R/c_L$ given by the proposed time integration method anf the central difference method. The numerical results of the proposed method is in the full agreement with theoretically predicted stress distributions [16]. The longitudinal and shear waves, reflected and Rayleigh's waves are generated in the disc and propagated with corresponding wave speeds. Based on the numerical results, the proposed time integration exhibits superior stress distributions without spurious oscillation for irregular (unstructured) meshes against the central difference method. In the central difference computation, the spurious oscillations and dispersion effects play a significant role and the obtained results are very polluted.

By this test, we demonstrated that accurate results of wave propagation problems on unstructured meshes can be acquired by the proposed time algorithm. To be successful in computations, favourable estimations of the critical time steps, Δt_L and Δt_S, are necessary to find. The main advantage of the proposed time integration method is that it is not necessary to determine the critical time steps exactly but only approximately.

6 CONCLUSIONS

In this paper, a step-by-step flowchart of a novel explicit three-time step algorithm in the predictor-predictor form for accurate FE computations of two- and three- dimensional wave propagation problems has been presented. Further, the algorithm mitigates spurious oscillations taking integrations of parts of equations of motion by each stability limits. With respect to the presented numerical test, the algorithm exhibits stress and strain distributions without front-shock and post-shock spurious oscillations. Generally, the algorithm improves results of FE modelling of wave propagation problems in solids. The submitted time algorithm is able to be easily implemented into a standard finite element code of general material and geometrical non-linear solid problems as dynamic plasticity or impact/contact problems.

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