

# ENHANCED FORMULA FOR A CRITICAL VELOCITY OF A UNIFORMLY MOVING LOAD INCLUDING SHEAR CONTRIBUTION

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**Key Words:** *Uniformly moving load, Critical velocity, Dynamic stiffness, Geometrical damping, Frequency-dependent foundation moduli.*

**Abstract.** The response of rails to moving loads has been a topic of interest for over a century. A related issue, the critical velocity of moving loads is an important matter related to track design. Analytically, in undamped environment, downward as well as upward displacements tend to infinity when the load is moving over an infinite rail at the critical velocity. However, the classical formula predicts such a critical velocity significantly overestimated than the one experienced in reality, indicating that the formula should be revised. In this contribution the dynamic equilibrium of the soil in the vertical direction is implemented to obtain two frequency dependent parameters incorporating the geometric damping and the soil mass inertia activated by induced vibrations. The new approach is tested on finite and infinite beams. Corrections to the classical formula are proposed.

## 1 INTRODUCTION

The response of rails to moving loads is of interest in the area of high-speed railway transportation. If simple geometries of the track and subsoil are considered, a theoretical concept that is based on the assumption that the track structure acts as a continuously supported beam (the rail) resting on a uniform layer of springs can be adopted. This layer of springs represents the underlying remainder of the track structure. The stiffness of such spring layer along the length of the track is named as the track modulus and defines Winkler's model. The Winkler model is often referred to as a "one-parameter model". Such a simplified model is traditionally used to estimate the critical velocity of moving trains.

The critical velocity of the load  $v_{cr}$  is defined as the phase velocity of the slowest free wave, but, in reality, this velocity should be compared to the Rayleigh-wave velocity of the ground [1] and therefore it is strange that the mass of the foundation is not accounted for in the classical formula.

## 2 TRACK STIFFNESS VERSUS TRACK MODULUS

The track is often characterized by two parameters: track stiffness,  $K$ , and track modulus,  $k$ . The track stiffness is defined as:

$$K = \frac{P}{w_{st}} \quad (1)$$

where  $P$  is the wheel load and  $w_{st}$  is the static deflection of the rail. Some realistic values based on experimental tests are presented in Table 1.

**Table 1:** Realistic values of the track stiffness

Type	$K$ (MN/m)
Soft platform	5-25
Clay platform	15-20
Platform with gravel	20-60
Rock platform	30-40
Ballast and hard platform	80-160

The track modulus is related to the track stiffness by:

$$k = \frac{K^{4/3}}{(64EI)^{1/3}} \quad (2)$$

based on the value of the static displacement of an infinite beam on an elastic foundation subjected to a static force  $P$ :

$$w_{st} = \frac{P}{2k} \sqrt[4]{\frac{k}{4EI}} \quad (3)$$

where  $EI$  stands for the beam bending stiffness. Thus the track modulus defines the Winkler constant of the one-parameter foundation model. Regarding the previous values in Table 1 and standard rail characteristics, the track modulus range from  $1\text{MN/m}^2$  to  $117\text{MN/m}^2$ . Values documented in the literature cover even larger interval from  $0.22\text{MN/m}^2$  to  $1000\text{MN/m}^2$ .

### 3 CRITICAL VELOCITY

The critical velocity of the load traversing an infinite Euler-Bernoulli beam on an elastic foundation is given by:

$$v_{cr}^{E-B} = \sqrt[4]{\frac{4kEI}{m^2}} \quad (4)$$

where  $E-B$  designates that Euler-Bernoulli theory is used and  $m$  stands for the beam mass per unit length. For a typical standard rail,  $EI=6.4\text{MNm}^2$ ,  $m=60\text{kg/m}$  and the lowest value of the track modulus introduced in the previous section,  $k=0.22\text{MN/m}^2$ , the critical velocity is above  $700\text{km/h}$ , which is not what has been observed in reality.

Formula (4) is closely related to a finite beam. Following [2, 3], the resonant velocity of a finite beam corresponds to the velocity for which the excitation frequency of the passing load equals to the corresponding beam natural frequency, thus:

$$v_{res} = \frac{L}{\lambda_j} \omega_j \quad (5)$$

where  $L$  is the beam length,  $\lambda_j / L$  is the wave number and  $\omega_j$  is the corresponding natural frequency. Such a resonant velocity can be attributed to each vibration mode. The critical velocity is the lowest resonant velocity. For a beam without an elastic foundation  $j_{cr}=1$  is always verified. When an elastic foundation is included, then one can consider the previous equation as function of  $j$  and establish the extreme value. For an Euler-Bernoulli beam it holds

$$v_{j_r}^{E-B} = \frac{L}{j_r \pi} \sqrt{\left(\frac{j_r \pi}{L}\right)^4 \frac{EI}{\mu} + \frac{k}{\mu}} \quad (6)$$

and the minimum value is achieved for a non-integer  $j_{cr}$ :

$$j_{cr} = \frac{L}{\pi} \sqrt[4]{\frac{k}{EI}} \quad (7)$$

Substituting  $j_{cr}$  back into Equation (6), Equation (4) is verified, as expected. Thus, the closest integer to  $j_{cr}$  indicates the critical velocity of the load passing over the finite Euler-Bernoulli beam. This value always overestimates the value related to the infinite beam. The main reason for incorrectness in the critical velocity predictions is the absence of the inertia effects of the foundation and no definition of the active foundation depth.

## 4 GENERALIZATIONS

Several studies were performed over the years in order to generalize Equation (4).

### 4.1 Beam level

By nullity condition imposed on the determinant of the dynamic stiffness matrix, for instance the value for the Timoshenko-Rayleigh beam can be obtained as [2]:

$$v_{cr}^{T-R} = \sqrt{\frac{1}{\mu(kr^2 - G\bar{A})^2} \left( k \left( EI(kr^2 - G\bar{A}) - 2r^2 G\bar{A}^2 \right) + 2G\bar{A} \sqrt{kG\bar{A}} \sqrt{kr^4 G\bar{A} - EI(kr^2 - G\bar{A})} \right)} \quad (8)$$

where  $r$  and  $G\bar{A}$  stand for the radius of gyration of the beam cross-section and for the shear stiffness of the beam, respectively,  $\bar{A}$  is the reduced (by Timoshenko's shear coefficient) cross-sectional area.

### 4.2 Soil level

Improvements of Winkler's model were obtained by introduction of another parameter in so-called Filonenko-Borodich, Pasternak or Hetenyi models (1954). This parameter can be explained as a shear contribution and thus removes the disadvantage of Winkler's springs that do not interact between themselves. It can equally be understood as a distributed rotational springs. This representation is easier to implement when finite element confirmation of

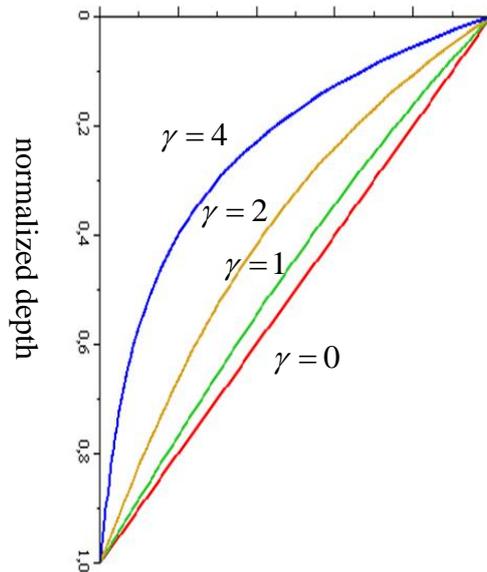
theoretical developments is required. The model is named as a “two-parameter model”. Other generalization is presented by Kerr (1965) as a “three-parameter model”. Also the Vlasov model (1966) is named as a “three-parameter model”, but the third parameter has different meaning as in the Kerr model. The Vlasov model improves the foundation model by introduction of the so called active depth of the soil. It is assumed that the deflection  $w$  varies inside the soil layer according to a function  $f(z)$  and  $w(x, y, z, t) = w(x, y, t)f(z)$ , where  $w(x, y, t)$  equals the deflection of the beam/soil contact point,  $x, y, z$  are spatial coordinates and  $t$  is the time. Then  $f(z)$  must verify  $f(0) = 1$  and  $f(H) = 0$ , where  $H$  is the active depth.  $f(z)$  can be expressed with the help of another parameter  $\gamma$  as:

$$f(z) = \frac{\sinh \left[ \gamma \left( 1 - \frac{z}{H} \right) \right]}{\sinh \gamma} \quad (9)$$

In the original development the parameter  $\gamma$  is arbitrary. A refinement of the Vlasov model is named as the “modified (refined) Vlasov model” (1988) introduced by Vallabhan and Das [4]. One of the possibilities of  $\gamma$  determination establishes a relation:

$$\left( \frac{\gamma}{H} \right)^2 = \frac{1 - 2\nu \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (\nabla w)^2 dx dy}{2(1 - \nu) \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} w^2 dx dy} \quad (10)$$

as this relation involves the still unknown vertical displacement  $w$ , an interactive procedure must be introduced in the solution. This model is named as the modified Vlasov model [4]. In Equation (10)  $\nu$  is the Poisson ration and  $\nabla$  is gradient operator.



**Figure 1:** Dependence of function  $f(z)$  on parameter  $\gamma$

The soil mass is added by a term assuming a linear distribution of function  $f(z)$  directly in the mass matrix of the structure. Analysing relation (9) it can be concluded that the only viable shapes of the function  $f(z)$  are contained within the region restricted by the linear distribution, as shown in Figure 1.

The two soil parameters are given by:

$$k = \int_0^H E^{oed} \left( \frac{df}{dz} \right)^2 dz = \frac{E^{oed}}{H} \frac{\gamma(\sinh \gamma \cosh \gamma + \gamma)}{2 \sinh^2 \gamma} \quad (11)$$

$$k_p = \int_0^H G f^2 dz = GH \frac{(\sinh \gamma \cosh \gamma - \gamma)}{2\gamma \sinh^2 \gamma} \quad (12)$$

where the oedometer modulus is  $E^{oed} = E(1-\nu) / ((1+\nu)(1-2\nu))$  and  $E$  and  $G$  stand for the Young and shear modulus of the soil, respectively. For linear distribution of function  $f(z)$ , i.e.  $\gamma = 0$  the previous parameters are given by

$$k = \frac{E^{oed}}{H}, \quad k_p = \frac{GH}{3} \quad (13)$$

## 5 DYNAMIC STIFFNESS

### 5.1 Generally

More correctly than by an additional term in the mass matrix, the mass inertia of the soil would be inserted directly in the soil moduli. Then the two soil parameters of the Pasternak model should be considered as frequency dependent,  $k(\omega)$  and  $k_p(\omega)$ . This will also remove the need of determination of an additional parameter  $\gamma$ . Function  $f(z)$  can be derived from the dynamic equilibrium of the soil in the vertical direction. Following [5]:

$$\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = \rho \frac{\partial^2 w}{\partial t^2} \quad (14)$$

where  $\sigma$  and  $\tau$  stand for normal and tangential stress components, respectively. The components of the deformation tensor are given by:

$$\varepsilon_z = w(x, y, t) \frac{df(z)}{dz}, \quad \gamma_{xz} = \frac{\partial w(x, y, t)}{\partial x} f(z), \quad \gamma_{yz} = \frac{\partial w(x, y, t)}{\partial y} f(z) \quad (15)$$

where  $\varepsilon$  and  $\gamma$  stand for the extension and engineering distortion, respectively. Therefore, the stress components can be expressed as:

$$\sigma_z = E^{oed} w(x, y, t) \frac{df(z)}{dz}, \quad \tau_{xz} = G \frac{\partial w(x, y, t)}{\partial x} f(z), \quad \tau_{yz} = G \frac{\partial w(x, y, t)}{\partial y} f(z) \quad (16)$$

Assuming harmonic vibrations and neglecting the shear stress derivatives, the differential equation for the function  $f(z)$  reads as:

$$\frac{d^2}{dz^2} f(z) + \lambda^2 f(z) = 0 \quad (17)$$

where the wave number  $\lambda$  is given by:

$$\lambda = \sqrt{\frac{\omega}{v_p}} = \sqrt[4]{\frac{\omega^2 \rho}{E^{oed}}} \quad (18)$$

and  $v_p$  is the velocity of the pressure waves. The solution of Equation (17) is:

$$f(z) = \cos \lambda z - \cotg \lambda H \sin \lambda z = \frac{\sin(\lambda(H-z))}{\sin \lambda H} \quad (19)$$

The total energy (both potential and kinetic) of the soil can be expressed as:

$$\begin{aligned} U &= \frac{1}{2} \int_{\Omega} \left\{ \int_0^H \left( E^{oed} \left( \frac{df}{dz} \right)^2 w^2 + Gf^2 \left( \left( \frac{\partial w}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 \right) - \omega^2 \rho f^2 w^2 \right) dz \right\} d\Omega \\ &= \frac{1}{2} \int_{\Omega} \left( kw^2 + k_p \left( \left( \frac{\partial w}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 \right) \right) d\Omega \end{aligned} \quad (20)$$

If a sufficiently extensive area  $\Omega$  is selected, the energy beyond this region can be neglected. In this formulation, the energy attributed to the Pasternak modulus in fact corresponds to the energy of distributed rotational springs. It follows:

$$k(\omega) = \int_0^H E^{oed} \left( \left( \frac{df}{dz} \right)^2 - (\lambda f)^2 \right) dz = \frac{E^{oed}}{H} \lambda H \frac{\cos \lambda H}{\sin \lambda H} \quad (21)$$

$$k_p(\omega) = \int_0^H Gf^2 dz = \frac{1}{2} GH \left( \frac{\lambda H - \sin \lambda H \cos \lambda H}{\lambda H \sin^2 \lambda H} \right) \quad (22)$$

and the vertical stress (the reaction pressure of the soil) at the contact is given by:

$$p_s(\omega) = k(\omega)w - k_p(\omega) \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) \quad (23)$$

If  $\lambda H$  tends to zero, static values of the Winkler and Pasternak parameters are verified as in Equation (13).

In summary, the effect of the viscoelastic foundation can be represented by the soil pressure, which for beam structures takes the following form:

$$p_s(\omega) = k(\omega)w - k_p(\omega) \frac{\partial^2 w}{\partial x^2} \quad (24)$$

## 5.2 Simply supported finite beam

In order to derive the new formula for the critical velocity, a finite beam on a frequency dependent foundation will be considered first. The governing equation of undamped free vibrations of the Euler-Bernoulli beam on a Pasternak foundation is given by [6]:

$$EI \frac{\partial^4 w}{\partial x^4} + m \frac{\partial^2 w}{\partial t^2} - k_p(\omega) \frac{\partial^2 w}{\partial x^2} + k(\omega) w = 0 \quad (25)$$

and the wave equation is:

$$EI p^4 - \omega^2 \mu - k_p(\omega) p^2 + k(\omega) = 0 \quad (26)$$

Considering a practical example, let the following values as specified in Table 2 be adopted.

**Table 2:** Numerical data in a practical example.

Property	Values
Beam bending stiffness $EI$ (MNm <sup>2</sup> )	6.4
Beam mass per unit length $m$ (kg/m)	60
Soil Young's modulus $E$ (MPa)	200
Soil Poisson's ratio $\nu$	0
Soil density $\rho$ (kg/m <sup>3</sup> )	2000
Beam length $L$ (m)	200
Active depth $H$ (m)	12

Because of the simple supports, the only beam deflection shape that verifies the boundary conditions is given by:

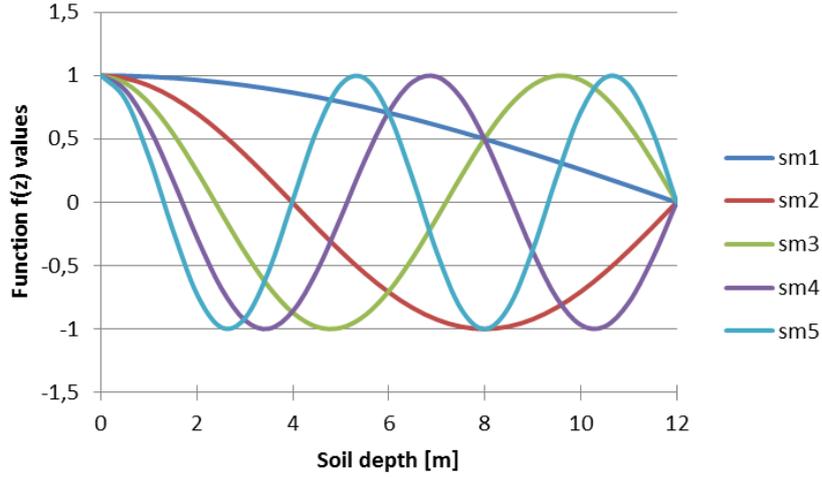
$$w_j(x) = \sin\left(\frac{j\pi}{L}x\right) \quad (27)$$

thus  $p = j\pi/L$  and natural frequencies can be determined. When the Pasternak contribution is omitted, there are infinite natural frequencies for a fixed  $j$  that have consecutive shapes of function  $f(z)$ . It can be concluded that the second soil mode frequency related to the fundamental beam mode shape will never be lower than any frequency of the first soil mode.

The first five soil modes related to the fundamental beam mode shape are presented in Figure 2. The contact condition

$$k = -E^{oed} \left. \frac{df(z)}{dz} \right|_{z=0} \quad (28)$$

is verified for the Winkler's term. Surprisingly this condition is not verified by the Vlasov model.



**Figure 2:** First five soil modes of the Winkler beam

Orthogonality conditions can be defined by:

$$\left( \mu + \tilde{\rho} \int_{z=0}^H f(\lambda_{j,k}) f(\lambda_{j,l}) dz \right) \int_{x=0}^L \sin^2 \left( \frac{j\pi}{L} x \right) dx = \delta_{kl} \quad \forall j, k \neq l \quad (29)$$

$$E^{oed} \int_{z=0}^H \frac{d}{dx} f(\lambda_{j,k}) \frac{d}{dx} f(\lambda_{j,l}) dz \int_{x=0}^L \sin^2 \left( \frac{j\pi}{L} x \right) dx + \int_{x=0}^L \sin \left( \frac{j\pi}{L} x \right) EI \frac{d^4}{dx^4} \sin \left( \frac{j\pi}{L} x \right) dx = \delta_{kl} \quad \forall j, k \neq l \quad (30)$$

where  $\tilde{\rho} = \rho b$  and  $b$  is the soil strip width. Then the modal mass is given by:

$$M_{j,k} = \int_{x=0}^L \left( \mu + \tilde{\rho} \int_{z=0}^H \left( \cos(\lambda_{j,k} z) - \frac{\cos \lambda_{j,k} H}{\sin \lambda_{j,k} H} \sin(\lambda_{j,k} z) \right)^2 dz \right) \sin^2 \left( \frac{j\pi}{L} x \right) dx \quad (31)$$

and the modal coordinate by a standard relation:

$$q_j(t) = \sum_k \frac{P}{M_{j,k} \omega_{j,k}^2 (1 - \Omega_{j,k}^2)} \left[ \sin(\Omega_{j,k} \omega_{j,k} t) - \Omega_{j,k} \sin(\omega_{j,k} t) \right], \quad \Omega_{j,k} = \frac{j\pi v}{L \omega_{j,k}} \quad (32)$$

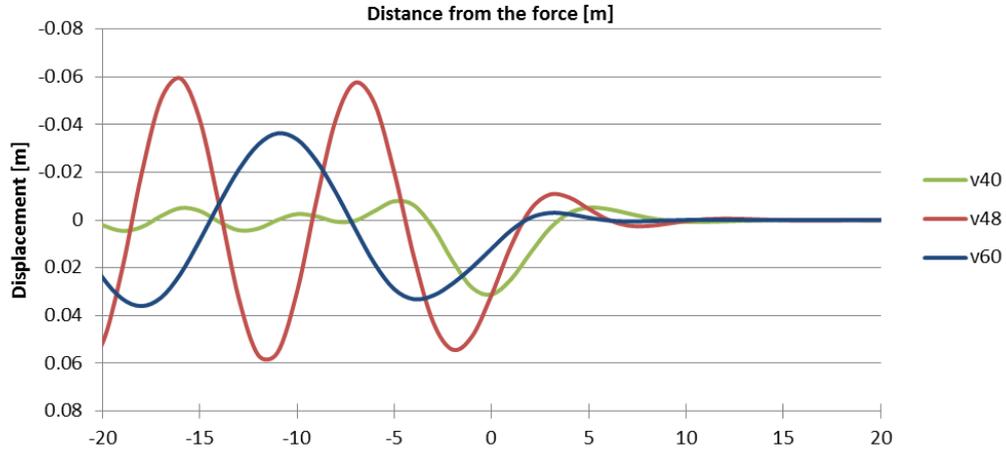
Then it is possible to determine the resonant velocities by standard procedures and consequently the new value of the critical velocity:

$$v_{enh}^{E-B} = \sqrt[4]{\frac{4k_{st} EI}{\tilde{\mu}^2}} \quad (33)$$

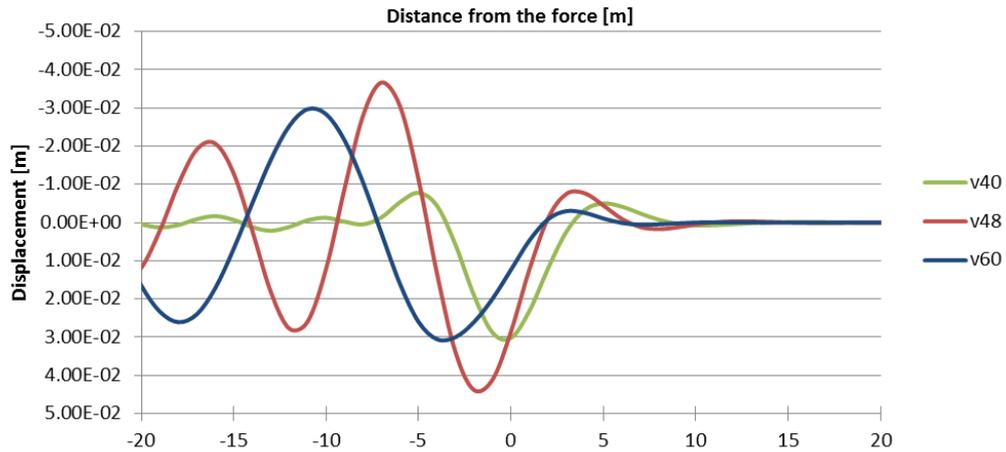
where

$$\tilde{\mu} = \mu + \zeta \rho b H, \quad \zeta = \frac{1}{H} \lim_{\lambda H \rightarrow \pi/2} \left( \int_{z=0}^H f^2(z) dz \right) \quad (34)$$

but a good approximation can be obtained with  $\zeta = 0.5$ . The change from subcritical to supercritical velocity is shown in Figure 3 and 4 for undamped and damped with  $\eta = 0.03$  cases.



**Figure 3:** Deflection of 400m length finite beam for several velocities around the critical one



**Figure 4:** Deflection of 400m length finite beam with 3% damping for several velocities around the critical one

If the Pasternak contribution is included, there are cases where the mode shapes do not represent a viable solution. This might be due to several simplifying assumptions along the determination of the wave equation.

### 5.3 Infinite beam

Regarding an infinite beam, Equation (25) can be represented in a moving coordinate system, now a damping term is included and the Pasternak term is omitted:

$$EI \frac{\partial^4 w}{\partial \tilde{x}^4} + mv^2 \frac{\partial^2 w}{\partial \tilde{x}^2} - cv \frac{\partial w}{\partial \tilde{x}} + k(\omega)w = P\delta(\tilde{x}), \quad k_{sr}H \frac{\partial^2 w}{\partial z^2} = \rho v^2 \frac{\partial^2 w}{\partial \tilde{x}^2} \quad (35)$$

By introduction of dimensionless characteristics

$$L_c = \frac{3\pi}{4} \sqrt[4]{\frac{4EI}{k_{st}}}, \bar{w} = \frac{w}{L_c}, \xi = \frac{\tilde{x}}{L_c}, \bar{P} = \left(\frac{3\pi}{2}\right)^2 \frac{P}{2\sqrt{k_{st}EI}}, \eta_b = \frac{c}{2\sqrt{k_{st}\mu}}, \zeta = \frac{z}{H} \quad (36)$$

it is obtained

$$\frac{d^4 \bar{w}}{d\xi^4} + \alpha^2 \left(\frac{3\pi}{2}\right)^2 \frac{d^2 \bar{w}}{d\xi^2} - \alpha \eta_b \left(\frac{3\pi}{2}\right)^3 \frac{d\bar{w}}{d\xi} + 4 \left(\frac{3\pi}{4}\right)^4 (1 + i\eta_f) \frac{\bar{\lambda}}{\tan \bar{\lambda}} \bar{w} = \bar{P} \delta(\xi) \quad (37)$$

after Fourier transformation the solution is given by:

$$\bar{w}^*(f) = \frac{\bar{P}}{(2\pi f)^4 - \alpha^2 \left(\frac{3\pi}{2}\right)^2 (2\pi f)^2 - \alpha \eta_b \left(\frac{3\pi}{2}\right)^3 (2\pi i f) + 4 \left(\frac{3\pi}{4}\right)^4 (1 + i\eta_f) \frac{\bar{\lambda}}{\tan \bar{\lambda}}} \quad (38)$$

There is infinite number of poles, and therefore a numerical solution by direct Fourier transform is presented in [7]. Nevertheless, it can be proven that the sum of all solution terms can be expressed analytically. It holds:

$$\bar{\lambda} = \frac{8}{3} \alpha \beta f \quad (39)$$

The final deflection shape can must be transformed from the complex space to the real one by:

$$w = \text{sign}(\text{Re}(w_c)) \sqrt{\text{Re}^2(w_c) + \text{Im}^2(w_c)} \cos\left(\text{atan}\left(\frac{\text{Im}(w_c)}{\text{Re}(w_c)}\right)\right) \quad (40)$$

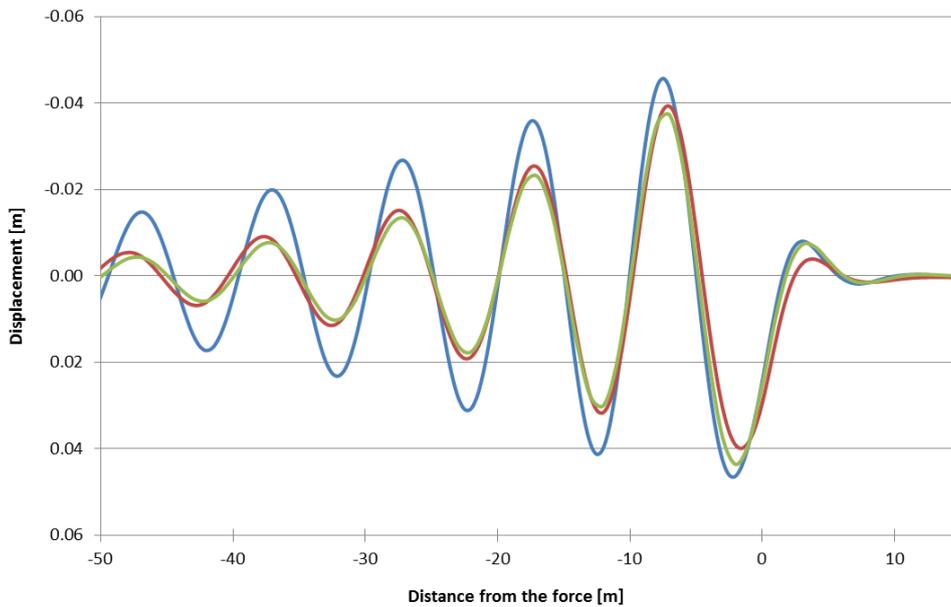


Figure 5: Deflection of an infinite beam for velocity of 49m/s

In Figure 5 deflection shapes are shown for  $\beta = 8.6$  and different approaches to express the damping term, namely  $\eta_b = 0.2$ ,  $\eta_f = 0.2$  and  $\eta = 0.03$ .

## 6 CONCLUSIONS

In this contribution the disadvantages of the standard formula for the critical velocity determination were summarized. The new approach that can improve the formula was introduced. This approach account for the mass inertia of the soil, directly by introduction of frequency dependent soil foundation parameters. Solution in an analytical form is presented for finite and infinite beams, critical velocity is determined and enhanced formula for the critical velocity is presented.

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