A 3-D MACH UNIFORM PRECONDITIONER FOR INCOMPRESSIBLE AND SUBSONIC FLOWS

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Abstract. In this paper, the detailed derivation and validation study of a recently developed three dimensional preconditioning formulation is presented.

1 INTRODUCTION

In recent years, beyond various different approaches, preconditioning methods gained increasing popularity for all speed flow solver development studies. Preconditioning methods employ special matrices which enhance the convergence behaviour when pre-multiplied with the time derivative terms. In literature, the pioneering preconditioning method is recognized to be the Artificial Compressibility Method (ACM) due to Chorin [1]. The first systematic approach on preconditioning methods is later conducted by Turkel [2] and a family of preconditioners are introduced. After these initial studies, several improvements and extensions are further developed. References [3] and [4] provide detailed review of the current preconditioning methods.

In the present preconditioning method, the Euler equations are non-dimensionalized similar to the pressure-splitting method of Merkle and Choi [5]. Furthermore, in addition to pressure, total energy is also split in a similar manner. In contrast to the preconditioning methods developed earlier, in the current approach the conservation of energy equation is preconditioned to enforce the divergence free constraint on the velocity field even at the limiting case of incompressible, zero Mach number flows.

The novel preconditioning method developed by the authors is already applied to 2-D flows [6]. In this study, the preconditioning method is extended to 3-D flows, it is validated and the preconditioned solutions are compared against non-preconditioned solutions of Euler equations on ONERA M6 test case for various Mach numbers for a performance assessment.

2 NON-DIMENSIONALIZATION

Non-dimensionalization is a standard process to reveal the general characteristics and relative importance of different terms of partial differential equations. In the non-dimensional Euler equations, two major problems, namely cancellation and eigenvalue disparity, are observed with the proper non-dimensionalization parameters. In this study the following non-dimensionalization is employed:

$$\rho^* = \frac{\rho}{\rho_\infty} \qquad u^* = \frac{u}{U_\infty} \qquad p^* = \frac{p}{\rho_\infty U_\infty^2} - P_r \qquad (\rho e_t)^* = \frac{(\rho e_t)}{\rho_\infty U_\infty^2} - (\rho e_t)_r$$
$$x^* = \frac{x}{x_r} \qquad t^* = \frac{tU_\infty}{x_r} \tag{1}$$

Where,

$$P_r = \frac{1}{\gamma M_\infty^2} \qquad (\rho e_t)_r = \frac{1}{\gamma(\gamma-1)M_\infty^2}$$

The non-dimensional conservation equations then become

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$
$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2 + p)}{\partial x} + \frac{\partial(\rho uv)}{\partial y} + \frac{\partial(\rho uw)}{\partial z} = 0$$
$$\frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho vu)}{\partial x} + \frac{\partial(\rho v^2 + p)}{\partial y} + \frac{\partial(\rho vw)}{\partial z} = 0$$
$$\frac{\partial(\rho w)}{\partial t} + \frac{\partial(\rho wu)}{\partial x} + \frac{\partial(\rho wv)}{\partial y} + \frac{\partial(\rho w^2 + p)}{\partial z} = 0$$
$$\frac{\partial(\rho e_t)}{\partial t} + \frac{\partial\left(u\left(\rho e_t + p + \frac{1}{(\gamma-1)M_\infty^2}\right)\right)}{\partial x} + \frac{\partial\left(v\left(\rho e_t + p + \frac{1}{(\gamma-1)M_\infty^2}\right)\right)}{\partial y} + \frac{\partial\left(w\left(\rho e_t + p + \frac{1}{(\gamma-1)M_\infty^2}\right)\right)}{\partial z} = 0 \tag{2}$$

The corresponding non-dimensional free-stream conditions are given below where asterisks are dropped for notational convenience.

$$\rho_\infty = 1 \qquad U_\infty = 1 \qquad p_\infty = 0 \qquad (\rho e_t)_\infty = 0.5$$

It should be noted that $\frac{1}{(\gamma-1)M_\infty^2}$ term grows and become dominant as the free stream Mach number decreases. This problem is known as "cancellation problem" in the literature.

The present non-dimensionalization also leads to the modified speed of sound given by

$$c = \sqrt{\frac{\gamma p}{\rho} + \frac{1}{\rho M_\infty^2}}$$

It should also be noted that as free stream Mach number goes to zero, the speed of sound goes to infinity which increases the differences between eigenvalues unboundedly. This problem is named as "eigenvalue disparity".

3 PRECONDITIONING

The present preconditioning method firstly relaxes the time derivative terms in both the conservation of mass and energy equations. It is achieved simply by multiplying the spatial derivative terms by M_∞^2 of these two equations.

$$\frac{\partial \rho}{\partial t} + M_\infty^2 \frac{\partial(\rho u)}{\partial x} + M_\infty^2 \frac{\partial(\rho v)}{\partial y} + M_\infty^2 \frac{\partial(\rho w)}{\partial z} = 0$$

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2 + p)}{\partial x} + \frac{\partial(\rho uv)}{\partial y} + \frac{\partial(\rho uw)}{\partial z} = 0$$

$$\frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho vu)}{\partial x} + \frac{\partial(\rho v^2 + p)}{\partial y} + \frac{\partial(\rho vw)}{\partial z} = 0$$

$$\frac{\partial(\rho w)}{\partial t} + \frac{\partial(\rho wu)}{\partial x} + \frac{\partial(\rho wv)}{\partial y} + \frac{\partial(\rho w^2 + p)}{\partial z} = 0$$

$$\frac{\partial(\rho e_t)}{\partial t} + \frac{\partial\left(u\left(M_\infty^2 \rho e_t + M_\infty^2 p + \frac{1}{(\gamma-1)}\right)\right)}{\partial x} + \frac{\partial\left(v\left(M_\infty^2 \rho e_t + M_\infty^2 p + \frac{1}{(\gamma-1)}\right)\right)}{\partial y} + \frac{\partial\left(w\left(M_\infty^2 \rho e_t + M_\infty^2 p + \frac{1}{(\gamma-1)}\right)\right)}{\partial z} = 0 \tag{3}$$

At the limiting case of zero free-stream Mach number, the equation set becomes

$$\frac{\partial \rho}{\partial t} = 0$$

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2 + p)}{\partial x} + \frac{\partial(\rho uv)}{\partial y} + \frac{\partial(\rho uw)}{\partial z} = 0$$

$$\frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho vu)}{\partial x} + \frac{\partial(\rho v^2 + p)}{\partial y} + \frac{\partial(\rho vw)}{\partial z} = 0$$

$$\frac{\partial(\rho w)}{\partial t} + \frac{\partial(\rho wu)}{\partial x} + \frac{\partial(\rho wv)}{\partial y} + \frac{\partial(\rho w^2 + p)}{\partial z} = 0$$

$$\frac{\partial(\rho e_t)}{\partial t} + \frac{1}{(\gamma-1)}\frac{\partial u}{\partial x} + \frac{1}{(\gamma-1)}\frac{\partial v}{\partial y} + \frac{1}{(\gamma-1)}\frac{\partial w}{\partial z} = 0 \tag{4}$$

In this preconditioned equation set, the conservation of mass equation diminishes whereas the conservation of energy equation becomes bounded and provides the divergence free velocity constraint on low Mach number flows. However, it is observed that the eigensystem of the equation set (3) is too complex to be useful. Time derivative of the ideal gas equation for incompressible flows is employed to obtain ACM formulation and benefit from its rather simple eigensystem.

$$\frac{1}{(\gamma-1)}\frac{\partial p}{\partial t} = \left(\frac{\partial(\rho e_t)}{\partial t} - u\frac{\partial(\rho u)}{\partial t} - v\frac{\partial(\rho v)}{\partial t} - w\frac{\partial(\rho w)}{\partial t}\right) \tag{5}$$

When equation (5) is considered, it is observed that subtraction of $\left(u\frac{\partial(\rho u)}{\partial t}+v\frac{\partial(\rho v)}{\partial t}+w\frac{\partial(\rho w)}{\partial t}\right)$ term from the conservation of energy equation recovers the original ACM formulation exactly at the incompressible flow limit.

The resulting preconditioning formulation becomes

$$\frac{\partial Q}{\partial t}+\Gamma\left(\frac{\partial E}{\partial x}+\frac{\partial F}{\partial y}+\frac{\partial G}{\partial z}\right)$$

$$Q=\begin{pmatrix}\rho\\ \rho u\\ \rho v\\ \rho w\\ \rho e\end{pmatrix}$$

$$E=\begin{pmatrix}\rho u\\ \rho u^2+P\\ \rho uv\\ \rho uw\\ u\left(\rho e+P+\frac{1}{(\gamma-1)M_\infty^2}\right)\end{pmatrix}\quad F=\begin{pmatrix}\rho v\\ \rho vu\\ \rho v^2+P\\ \rho vw\\ v\left(\rho e+P+\frac{1}{(\gamma-1)M_\infty^2}\right)\end{pmatrix}\quad G=\begin{pmatrix}\rho w\\ \rho wu\\ \rho wv\\ \rho w^2+P\\ w\left(\rho e+P+\frac{1}{(\gamma-1)M_\infty^2}\right)\end{pmatrix}$$

$$\Gamma=\begin{bmatrix}M_\infty^2 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & (1-M_\infty^2)u & (1-M_\infty^2)v & (1-M_\infty^2)w & M_\infty^2\end{bmatrix} \tag{6}$$

4 FLOW SOLVER

The Mach uniform preconditioner developed is implemented in a three dimensional parallel in-house flow solver. The flow solver employs Runge-Kutta temporal discretization scheme and the first order Roe’s approximate Riemann solver for the convective flux evaluations. The Roe fluxes are evaluated with the general formula

$$\frac{1}{2}(F_L+F_R-T|\lambda|T^{-1}\Delta Q)$$

Where F_L and F_R are left and right state fluxes respectively. λ and T are eigenvalues and the eigenvectors of the preconditioned jacobian matrix respectively.

$$\lambda=\begin{bmatrix}u\\ u\\ M_\infty^2 u\\ u-c\\ u+c\end{bmatrix}$$

$$T = \begin{bmatrix} 0 & 0 & 1 & M_\infty^2 & M_\infty^2 \\ 0 & 0 & u & u-c & u+c \\ w & w & 0 & \frac{vF}{c} & -\frac{vG}{c} \\ v & -v & 0 & \frac{wF}{c} & -\frac{wG}{c} \\ 2vw & 0 & \frac{u^2-v^2-w^2}{2}+\frac{(M_\infty^2-1)u^2}{\gamma-1} & D-E & D+E \end{bmatrix}$$

$$T^{-1} = \begin{bmatrix} -\frac{(v^2+w^2)q_t(\gamma-1)}{4vwc^2} & \frac{u(v^2+w^2)H}{2vwc^2} & \frac{(\gamma-1)(v^2+w^2)+c^2}{2wc^2} & \frac{(\gamma-1)(v^2+w^2)+c^2}{2vc^2} & -\frac{(\gamma-1)(v^2+w^2)}{2vwc^2} \\ -\frac{(v^2-w^2)q_t(\gamma-1)}{4vwc^2} & \frac{u(v^2-w^2)H}{2vwc^2} & \frac{(\gamma-1)(v^2-w^2)+c^2}{2wc^2} & \frac{(\gamma-1)(v^2-w^2)-c^2}{2vc^2} & -\frac{(\gamma-1)(v^2-w^2)}{2vwc^2} \\ \frac{J}{2FG} & -\frac{M_\infty^2(\gamma-1)u}{FG} & -\frac{M_\infty^2(\gamma-1)v}{FG} & -\frac{M_\infty^2(\gamma-1)w}{FG} & \frac{M_\infty^2(\gamma-1)}{FG} \\ \frac{(\gamma-1)q_t+2uc}{4cF} & -\frac{Hu+c}{2cF} & -\frac{(\gamma-1)v}{2cF} & -\frac{(\gamma-1)w}{2cF} & \frac{(\gamma-1)}{2cF} \\ \frac{(1-\gamma)q_t+2uc}{4cG} & \frac{Hu-c}{2cG} & \frac{(\gamma-1)v}{2cG} & \frac{(\gamma-1)w}{2cG} & -\frac{(\gamma-1)w}{2cG} \end{bmatrix}$$

$$q_t = u^2+v^2+w^2$$

$$c = \sqrt{\frac{M_\infty^2\gamma p+1}{\rho}+(1-M_\infty^2)u}$$

$$D = \frac{c^2}{(\gamma-1)} - \frac{(M_\infty^2-2)q_t}{2} + \frac{(M_\infty^2-1)u^2}{(\gamma-1)}$$

$$E = uc - \frac{uv^2(M_\infty^2-1)}{c} - \frac{uw^2(M_\infty^2-1)}{c}$$

$$F = (M_\infty^2-1)u+c$$

$$G = (M_\infty^2-1)u-c$$

$$H = (M_\infty^2+\gamma-2)$$

$$J = M_\infty^2(\gamma-1)q_t - 2u^2(M_\infty^2-1) - 2c^2$$

In this study, the following standard boundary conditions are employed. These boundary conditions are the same as the boundary conditions employed in ACM formulations.

$$\rho = \rho_\infty \quad \rho u = \rho u_\infty \quad \rho v = \rho v_\infty \quad \rho w = \rho w_\infty \quad \text{p}= p_{ext} \quad \textit{for inflow}$$

$$\rho = \rho_{ext} \quad \rho u = \rho u_{ext} \quad \rho v = \rho v_{ext} \quad \rho w = \rho w_{ext} \quad \text{p}= p_\infty \quad \textit{for outflow}$$

5 RESULTS AND DISCUSSION

5.1 Validation

Transonic flow over ONERA M6 wing is selected for both validation and performance evaluation cases. ONERA M6 wing has a relatively simple geometry with no twist and a symmetrical airfoil. This test case is selected because of its wide usage as a standard test case for the CFD codes developed. Among various free stream Mach numbers and angle of attacks, Test 2308 of reference [6] is selected where $M_\infty = 0.8395$ and $\propto = 3.06$.

For this purpose, a 3-D unstructured mesh with 447842 cells and 91418 nodes is generated and used.

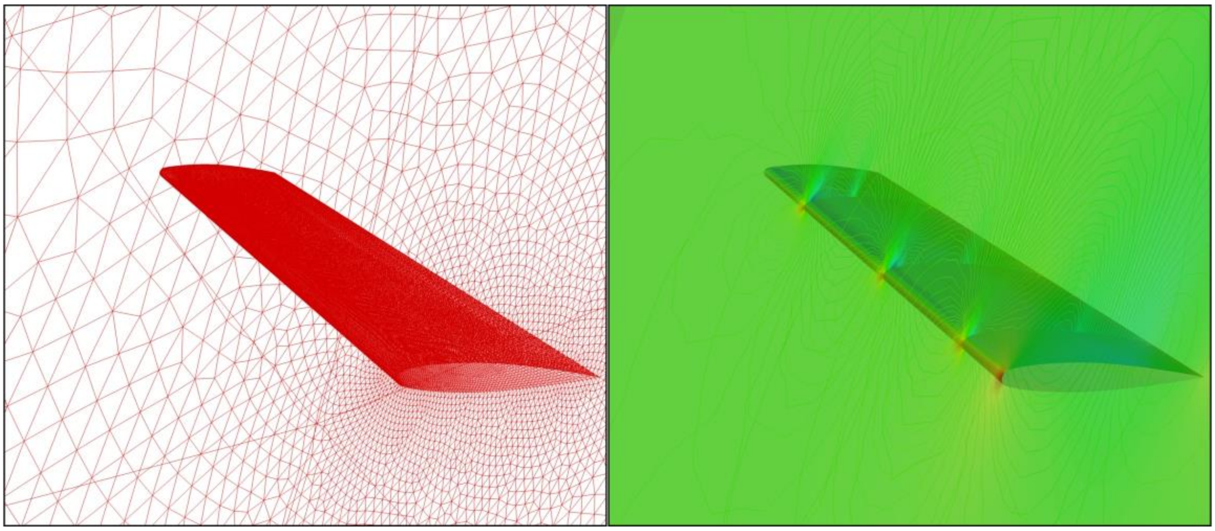

Figure 1: Computational mesh and pressure distribution for flow over ONERA M6 wing at $M_\infty = 0.8395$ and $\propto = 3.06$

In **Figure 2** the distribution of pressure coefficient over two different spanwise locations are compared against the experimental data. As expected, the preconditioned and non-preconditioned solutions are the same at both stations and the inviscid flow prediction is in better agreement with the experimental data at station $\frac{y}{b} = 0.2$. The disagreement at station $\frac{y}{b} = 0.95$ is attributed to the viscous effects due to tip vortex.

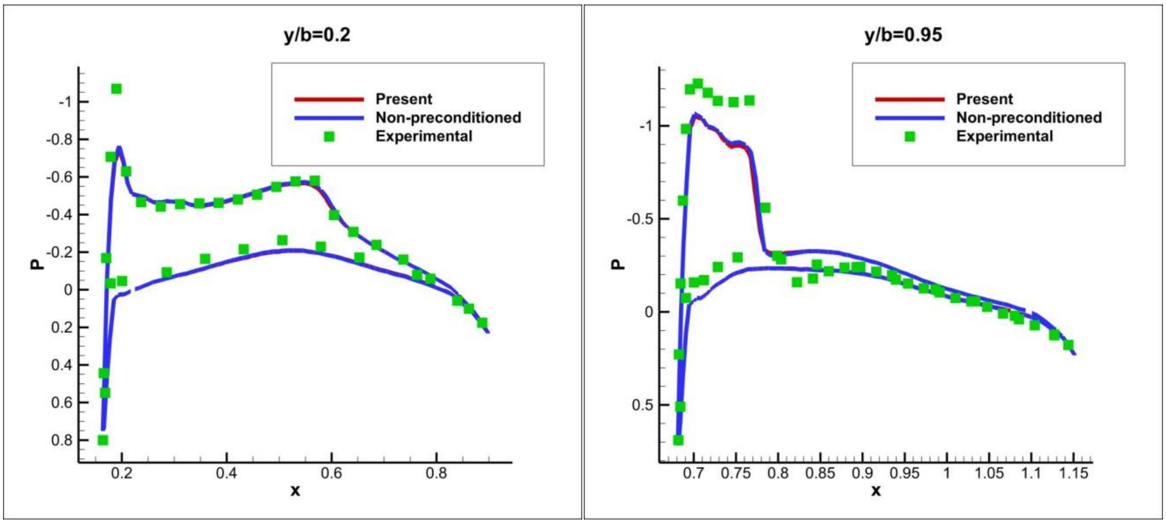

Figure 2: Pressure distribution on $\frac{\mathrm{y}}{\mathrm{b}} = 0.2$ and $\frac{\mathrm{y}}{\mathrm{b}} = 0.95$ for flow over ONERA M6 wing at $M_\infty = 0.8395$ and $\propto = 3.06$

5.2 Performance Evaluation

After the validation of the formulation developed, its impact on the convergence characteristics is analyzed on a range of free stream Mach numbers from incompressible to transonic speeds. In all the cases, the CFL value is kept at its highest values while the solution stability is achieved. It is observed that, while maximum CFL values occur to be around 3 for all preconditioned cases, CFL values as low as 0.01 are needed for non-preconditioned solutions. The pressure coefficient distribution and the residual histories are presented in **Figure 3** to **Figure 7** where 0.5m span location is selected for comparison.

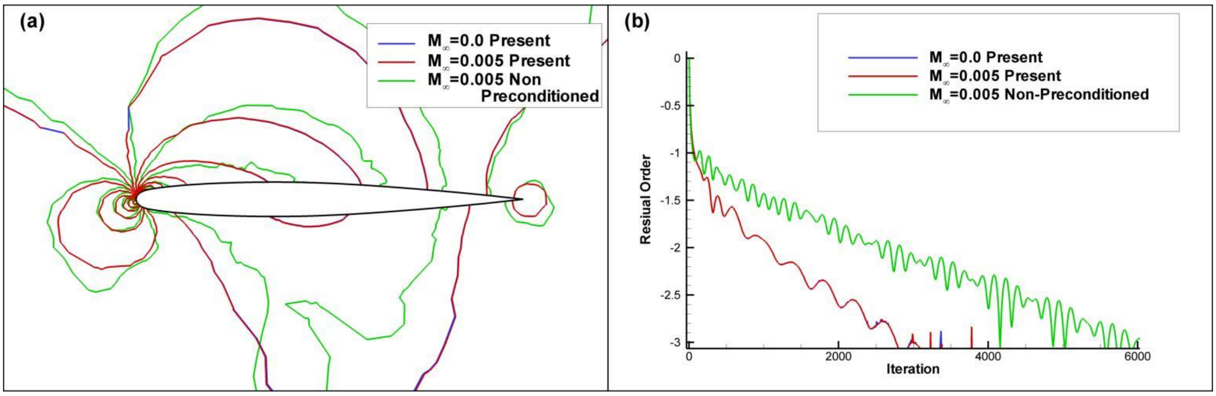

Figure 3: (a) Pressure distributions and (b) residual histories for flow over ONERA M6 wing at $M_\infty = 0.0$ and $M_\infty = 0.005$ and $\propto = 3.06$

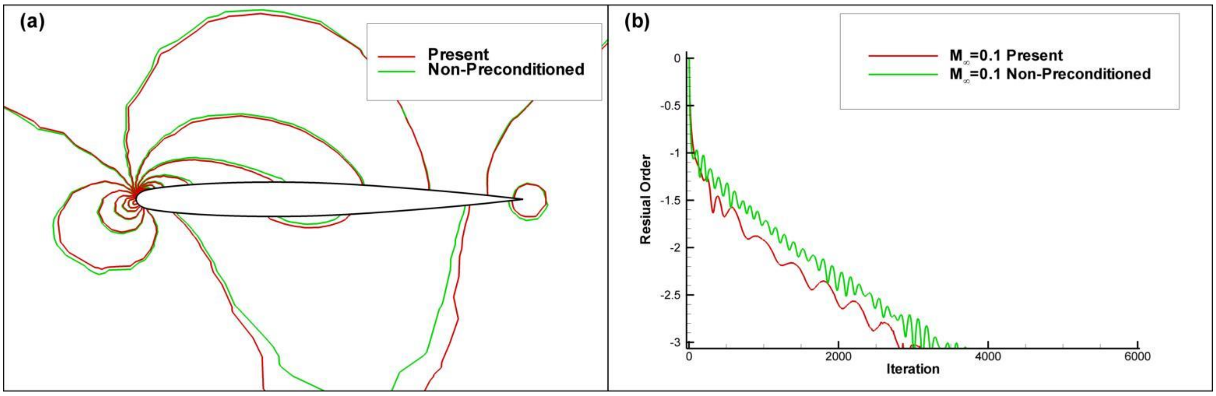

Figure 4: (a) Pressure distributions and (b) residual histories for flow over ONERA M6 wing at $M_\infty = 0.1$ and $\propto = 3.06$

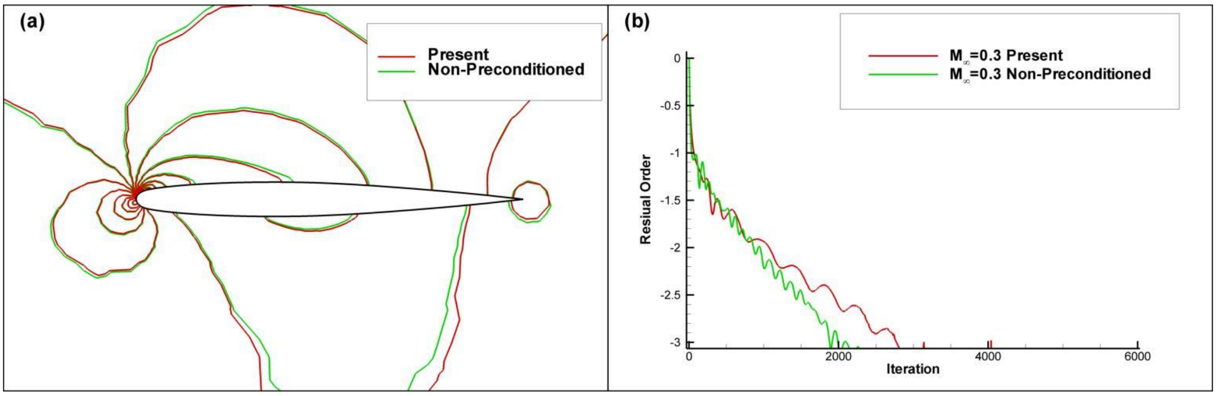

Figure 5: (a) Pressure distributions and (b) residual histories for flow over ONERA M6 wing at $M_\infty = 0.3$ and $\propto = 3.06$

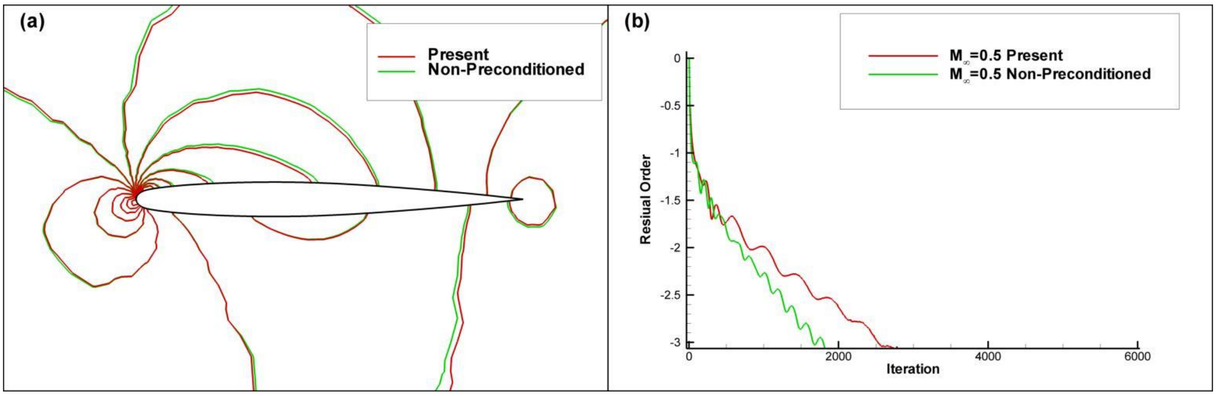

Figure 6: (a) Pressure distributions and (b) residual histories for flow over ONERA M6 wing at $M_\infty = 0.5$ and $\propto = 3.06$

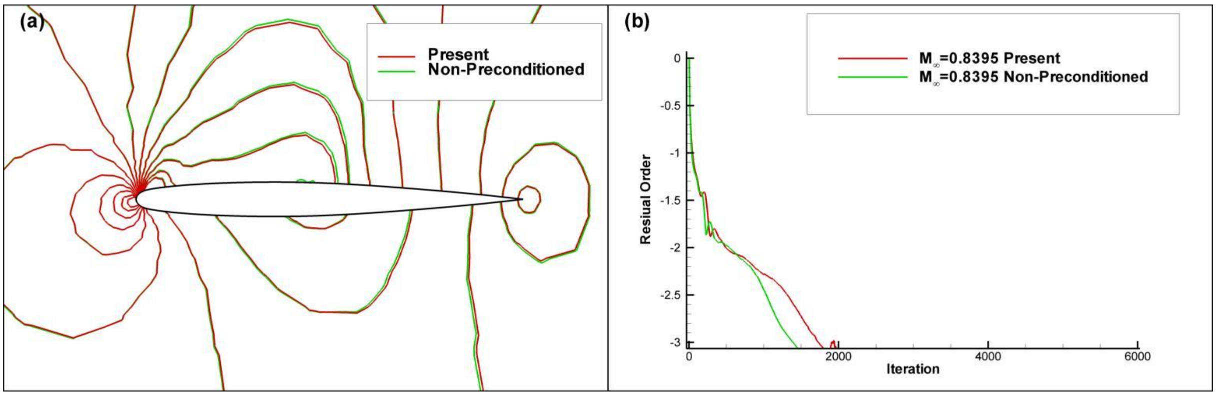

Figure 7: (a) Pressure distributions and (b) residual histories for flow over ONERA M6 wing at $M_\infty = 0.8395$ and $\propto = 3.06$

It is observed from the results and residual graphs that the present preconditioning matrix has a comparable accuracy and convergence rate with non-preconditioned formulation for all flow conditions except $M_\infty = 0.005$. Slight differences are attributed to reflecting boundary conditions.

At case $M_\infty = 0.005$ condition, the non-preconditioned solution is obtained by employing a CFL value of 0.01 and a stable solution cannot be achieved as shown in **Figure 3** whereas the preconditioned formulation do not exhibit any deterioration on convergence rate. $M_\infty = 0.005$ is the smallest Mach number for which a non-preconditoned solution can be obtained. All of the preconditioned solution residuals are also given on **Figure 8**. This figure proves the Mach uniform convergence of preconditioned formulation up to $M_\infty = 0.5$.

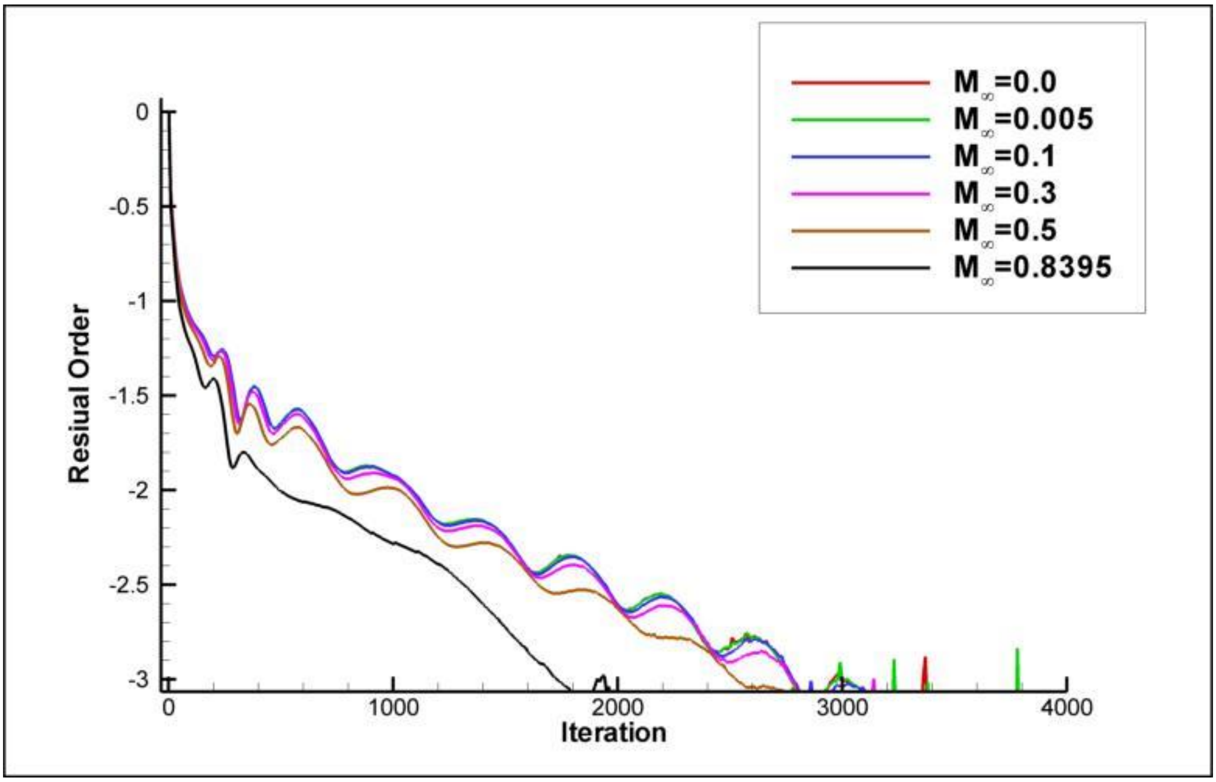

Figure 8: Residual histories for flow over ONERA M6 at $\propto = 3.06$

12 CONCLUSIONS

In this study, the Mach uniform preconditioning method developed earlier is extended to 3-D flows. The preconditioned equations provide stability and Mach uniform convergence characteristics for very low subsonic flows including zero free-stream Mach number. The developed 3-D formulation is validated for flow over ONERA M6 wing at several free stream Mach number flows and the instability of the compressible flow solvers at very low Mach numbers are successfully prevented.

REFERENCES

[1] Chorin A.J. A numerical method for solving incompressible viscous problems. *Journal of Computational Physics* (1967) **2**:12-26.
[2] Turkel E. Preconditioning methods for solving the incompressible and low speed compressible equations. *Journal of Computational Physics* (1987) **72**:277-298.
[3] Turkel E. Preconditioning techniques in computational fluid dynamics *Annual Reviews in*

Fuid Dynamics (1999) **31**:385-416.

[4] Turkel E. and Vatsa V.N. Local preconditioners for steady state and dual time-stepping, mathematical modeling and numerical analysis. *ESAIM: M2AN* (2005) **39(3)**:515-536.

[5] Bas O., Tuncer I. H. and Kaynak U. A Mach uniform preconditioner for incompressible and subsonic flows. *Int. J. Num. Meth. Fluids* (2014) **74**:100-112.

[6] Schmitt V. and Charpin F. *Pressure distributions on the ONERA-M6 wing at transonic Mach numbers*. AGARD AR 138 (1979).