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STRESS FORMULATION OF COMPLEX VARIABLE BOUNDARY INTEGRAL EQUATION FOR SOLVING TORSION PROBLEMS

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Abstract. Theory of complex variables is a very powerful mathematical technique for solving two-dimensional problems satisfying the Laplace equation. Based on the Cauchy integral formula, the complex variable boundary integral equation (CVBIE) can be constructed. However, the limitation of the above CVBIE is only suitable for holomorphic (analytic) functions. To solve a harmonic-function pair without satisfying the Cauchy-Riemann equations, we propose a new CVBIE that can be employed to solve any harmonic function in two-dimensional Laplace problems. We can derive the present CVBIE by using the Borel-Pompeiu formula. The difference between the present CVBIE and the conventional CVBIE is that the former one has two boundary integrals instead of only one boundary integral is in the latter one. When the unknown field is a holomorphic (analytic) function, the present CVBIE can be reduced to the conventional CVBIE. To examine the present CVBIE, we consider a torsion problem in this paper since the two shear stress fields satisfy the Laplace equation but do not satisfy the Cauchy-Riemann equations. Based on the present CVBIE, we can straightforward solve the stress fields and the torsional rigidity simultaneously. Finally, several examples, circular bar, elliptical bar, equilateral triangular bar, rectangular bar, asteroid bar and circular bar with keyway, were demonstrated to check the validity of the present method.

1 INTRODUCTION

For many engineering problems, their physical phenomena can be described by certain mathematical models such as Laplace, Helmholtz, biharmonic or biHelmholtz equation etc. For instance, steady-state heat conduction problems [1, 2], electrostatic potential [3], torsion problems [4], and potential flow problems [5] satisfy the Laplace equation, membrane vibration [6], water wave problems [7], acoustics [8], electromagnetic radiation [9] and seismology [10] are simulated by the Helmholtz equation while Stokes's flow [11] and plate vibration [12] are governed by the biharmonic and biHelmholtz equations, respectively. To simulate their physical behavior in advance, we must solve the corresponding mathematical model. For this reason, researchers and engineers paid more attention to develop various kinds of numerical methods such as the finite difference method (FDM), the finite difference method (FEM) the boundary element method (BEM) and meshless methods, etc.

Although the FEM is one of the most popular methods, it costs time on constructing the geometry model and needs to generate the mesh over the whole domain. In recent years, the BEM is an alternative approach to solve engineering problems. It is more efficient than the FEM since it is a mesh reduction method and only boundary discretization is required. There are many books and literatures focusing on the BEM such as Banerjee [13], Beskos [14], Bonnet [15], Brebbia [16], Chen and Hong [17], Crouch and Starfield [18], Kytbe [19], Liggett and Liu [20], and Mukherjee [21]. While the BEM is a well-developed numerical approach for solving engineering problems with general geometries, it results in errors of geometric discretization, boundary contour integrals and ill-posedness due to the fully populated influence matrix.

Regarding the above researchers, they paid more attention on the BEM and BIE in the real variable space. For the complex variable boundary element method (CVBEM), it has been applied in many fields. Hromadka II and Lai [22] developed a theorem of CVBEM to solve engineering problems. The CVBEM [23,24] is based on the Cauchy integral formula, residue theorem and Cauchy-Riemann equation in the complex analysis. It is more efficient for solving two-dimensional Laplace problems and plane elasticity than using the real variables boundary element method (RVBEM) since a complex-variable function can contain two fields. Nevertheless, it still has a limitation. The CVBEM based on the Cauchy integral formula is only suitable for the analytic (holomorphic) functions. When the two unknown fields are not the Cauchy-Riemann equation pairs, the conventional one can not be used to solve any harmonic functions such as the two shear stress fields in Saint-Venant's torsion problem. Accordingly, Di Paola et. al. [25] transformed the two shear stress fields to satisfy the Cauchy-Riemann equation by adding some terms and employed the line element-less method to solve it. Later, Barone and Pirrotta [26] used the complex polynomial method proposed by Hromadka and Guymon [27] to solve the same problem. Therefore, the conventional CVBEM can be employed to solve the two shear stress fields simultaneously. The corresponding complex-valued function is still an analytic (holomorphic) function.

In this paper, we propose a new CVBEM that can be utilized to solve any harmonic functions in the two dimensional domain. We derive the generalized Cauchy integral formula in terms of the form boundary integral by using the Borel-Pompeiu formula [28]. Not only the analytic (holomorphic) functions but also the harmonic functions satisfy the general one. Therefore, the corresponding complex variable boundary integral equation (CVBIE) can be derived. To check the validity the present CVBEM, the two shear stress fields of Saint-Venant's torsion problem were considered.

2 PROBLEM STATEMENT

Here we consider an elastic bar subjected to the pure torsion at the end plane. According to

Saint-Venant's torsion theory, the displacement field on the end plane can be assumed as shown below:

$$u = -\alpha yz, \tag{1}$$

$$v = \alpha xz, \tag{2}$$

$$w = \alpha \varphi(z), \tag{3}$$

where α is the twist angle of per unit length of the bar and $\varphi(z)$ is the warping function. By substituting the displacement field of Eqs. (1)-(3) to the strain-displacement relations, we have

$$\varepsilon_x = \varepsilon_y = \varepsilon_z = \gamma_{xy} = 0, \tag{4}$$

$$\gamma_{yz} = \frac{\partial \varphi}{\partial y} + \alpha x, \tag{5}$$

$$\gamma_{xz} = \frac{\partial \varphi}{\partial x} - \alpha y. \tag{6}$$

Then employing the stress-strain relations, we can obtain the stress field as shown below:

$$\sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0, \tag{7}$$

$$\tau_{yz} = G\left(\frac{\partial \varphi}{\partial y} + \alpha x\right), \tag{8}$$

$$\tau_{xz} = G\left(\frac{\partial \varphi}{\partial x} - \alpha y\right), \tag{9}$$

where G is the shear modulus. By applying the compatibility and equilibrium equations for stress field in Eqs. (7) to (9), we have

$$-\frac{\partial \tau_{yz}}{\partial x} + \frac{\partial \tau_{xz}}{\partial y} = -2G\alpha, \tag{10}$$

$$\frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{xz}}{\partial x} = 0. \tag{11}$$

By taking the process of $\frac{\partial(11)}{\partial y} - \frac{\partial(10)}{\partial x}$, we have

$$\frac{\partial^2 \tau_{yz}}{\partial y \partial y} + \frac{\partial^2 \tau_{xz}}{\partial x \partial y} + \frac{\partial^2 \tau_{yz}}{\partial x \partial x} - \frac{\partial^2 \tau_{xz}}{\partial y \partial x} = \frac{\partial^2 \tau_{yz}}{\partial x^2} + \frac{\partial^2 \tau_{yz}}{\partial y^2} = 0. \tag{12}$$

Similarly, by taking the process of $\frac{\partial(10)}{\partial y} + \frac{\partial(11)}{\partial x}$, we have

$$-\frac{\partial^2 \tau_{yz}}{\partial x \partial y}+\frac{\partial^2 \tau_{xz}}{\partial y \partial y}+\frac{\partial^2 \tau_{yz}}{\partial y \partial x}+\frac{\partial^2 \tau_{xz}}{\partial x \partial x}=\frac{\partial^2 \tau_{xz}}{\partial x^2}+\frac{\partial^2 \tau_{xz}}{\partial y^2}=0. \tag{13}$$

According to Eqs. (12) and (13), it is found that the two shear stress fields τ_{yz} and τ_{xz} are the harmonic functions. In this paper, we intend to employ a complex variable boundary integral equation (CVBIE) to solve the stress fields straightforward by setting the complex function as shown below:

$$f(z)=\tau_{yz}+i\tau_{xz}. \tag{14}$$

However, the above complex function does not satisfy the Cauchy-Riemann equations since the compatibility equation in Eq. (10) is not equal to zero. It is a harmonic function but not a holomorphic (analytic) function. That is to say, τ_{yz} and τ_{xz} do not satisfy the Cauchy-Riemann pair. The set of harmonic functions includes the set of holomorphic functions and their relationship is shown in Fig. 1. For this reason, the conventional CVBIE based on the Cauchy integral formula is not suitable for solving any harmonic function. In this paper, we propose a new CVBIE that can be employed to solve all harmonic functions. Since no external force acting on the lateral surface of the bar, we have the traction free boundary condition as given below:

$$(\tau_{xz},\tau_{yz})\cdot(n_x,n_y)=\tau_{xz}n_x+\tau_{yz}n_y=0, \tag{15}$$

where n_x and n_y are horizontal and vertical components of the unit outward normal vector, respectively. The static equivalence condition for the torque M is defined as:

$$M=\iint_\Omega(-y\tau_{xz}+x\tau_{yz})dxdy, \tag{16}$$

where Ω stands for the domain of the cross section. Besides, we can obtain the second boundary condition in terms of the complex function $f(z)$ from Eqs. (10) and (11) as shown below:

$$\frac{\partial f(z)}{\partial \bar{z}}=G\alpha. \tag{17}$$

3 BOUNDARY INTEGRAL EQUATIONS IN COMPLEX VARIABLES

In this section, we derive the new CVBIE from the Borel-Pompeiu formula [28]. First, we revisit the Borel-Pompeiu formula. The Gauss theorem for the two-dimensional case is

$$\int_\Omega \nabla\cdot w\,dA=\int_{\partial\Omega} w\cdot n\,dS, \tag{18}$$

where ∇ is the gradient operator and w is a real-valued function. In the complex analysis, we can obtain the complex-valued form of Gauss theorem from the above equation as

$$\int_\Omega \frac{\partial w(s)}{\partial \bar{s}}ds_x ds_y=\frac{1}{2i}\int_{\partial\Omega} w(s)\,ds, \tag{19}$$

and its conjugate form is

$$\int_{\Omega} \frac{\partial w(s)}{\partial s} ds_x ds_y = -\frac{1}{2i} \int_{\partial\Omega} w(s)\, d\bar{s}, \tag{20}$$

where $s = s_x + is_y$ and $w(s)$ is a complex-valued function. By substituting $w(s) = \dfrac{f(s)}{s-z}$ into Eq. (19), we have the Borel-Pompeiu formula for $z \in \mathbf{C} \setminus \bar{\Omega}$ as

$$0 = \frac{1}{2i} \int_{\partial\Omega} \frac{f(s)}{s-z} ds - \int_{\Omega} \frac{1}{s-z} \frac{\partial f(s)}{\partial \bar{s}} ds_x ds_y. \tag{21}$$

Then, substituting $\dfrac{\partial f(s)}{\partial \bar{s}} \ln|s-z|^2$ for $w(s)$ in Eq. (20), we have

$$-\int_{\Omega} \frac{1}{s-z} \frac{\partial f(s)}{\partial \bar{s}} ds_x ds_y = \frac{1}{2i} \int_{\partial\Omega} \frac{\partial f(s)}{\partial \bar{s}} \ln|s-z|^2\, d\bar{s} + \int_{\Omega} \frac{\partial^2 f(s)}{\partial \bar{s}\,\partial s} \ln|s-z|^2\, ds_x ds_y, \tag{22}$$

where $f(s) \in C^1(\bar{\Omega}) \cap C^2(\Omega)$. By substituting Eq. (21) into Eq. (22), we have

$$0 = \frac{1}{2i} \int_{\partial\Omega} \frac{f(s)}{s-z} ds + \frac{1}{2i} \int_{\partial\Omega} \frac{\partial f(s)}{\partial \bar{s}} \ln|s-z|^2\, d\bar{s} + \int_{\Omega} \frac{\partial^2 f(s)}{\partial \bar{s}\,\partial s} \ln|s-z|^2\, ds_x ds_y. \tag{23}$$

If $f(s)$ satisfies the two-dimensional Laplace equation $\dfrac{\partial^2 f(s)}{\partial \bar{s}\,\partial s} = 0$, *i.e.*, $f(s)$ is a harmonic function, the area integral in Eq. (23) vanishes. Therefore, we have

$$0 = \frac{1}{2i} \int_{\partial\Omega} \frac{f(s)}{s-z} ds + \frac{1}{2i} \int_{\partial\Omega} \frac{\partial f(s)}{\partial \bar{s}} \ln|s-z|^2\, d\bar{s}. \tag{24}$$

When $z \in \Omega$, a singular point exists in the domain Ω in Eq. (24). To deal with this problem to obtain the singular BIE for $z \in \Omega$, we need to employ the limiting process. By this way, we obtain

$$f(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(s)}{s-z} ds + \frac{1}{2\pi i} \int_{\partial\Omega} \frac{\partial f(s)}{\partial \bar{s}} \ln|s-z|^2\, d\bar{s},\ z \in \Omega. \tag{25}$$

If the field point z is outside the domain ($z \in \mathbf{C} \setminus \bar{\Omega}$), we can also obtain the null field BIE from Eq. (25) as

$$0 = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(s)}{s-z} ds + \frac{1}{2\pi i} \int_{\partial\Omega} \frac{\partial f(s)}{\partial \bar{s}} \ln|s-z|^2\, d\bar{s},\ z \in \mathbf{C} \setminus \bar{\Omega}. \tag{26}$$

If the field point z is located on the boundary $(z \in \partial\Omega)$, Eq. (26) yields the singularity since $z = s$ may occur. Therefore, we also need to employ the limiting process and introduce the concept of Cauchy principal value. Then, we have

$$\frac{\alpha}{2\pi} f(z) = \frac{1}{2\pi i} C.P.V. \int_{\partial\Omega} \frac{f(s)}{s-z} ds + \frac{1}{2\pi i} \int_{\partial\Omega} \frac{\partial f(s)}{\partial \bar{s}} \ln|s-z|^2\, d\bar{s},\ z \in \partial\Omega, \tag{27}$$

where α is the solid angle and the $C.P.V.$ is the Cauchy principal value. After arranging Eqs. (25) to (27), we have the complex variable singular BIE,

$$c(z)f(z)=\frac{1}{2\pi i}\int_{\partial\Omega}^{(z)}\frac{f(s)}{s-z}ds+\frac{1}{2\pi i}\int_{\partial\Omega}\frac{\partial f(s)}{\partial\bar{s}}\ln\left|s-z\right|^{2}d\bar{s}, \tag{28}$$

where

$$c(z)=\begin{cases}1, & z\in\Omega,\\ \dfrac{\alpha}{2\pi}, & z\in\partial\Omega,\\ 0, & z\in\mathbf{C}\setminus\bar{\Omega},\end{cases} \tag{29}$$

and

$$\int_{\partial\Omega}^{(z)}=\begin{cases}\int_{\partial\Omega}, \; z\in\Omega,\\ C.P.V.\int_{\partial\Omega}, & z\in\partial\Omega,\\ \int_{\partial\Omega}, & z\in\mathbf{C}\setminus\bar{\Omega}.\end{cases} \tag{30}$$

4 DISCRETIZATION OF THE COMPLEX VARIABLE BOUNDARY INTEGRAL EQUATION FOR SOLVING THE STRESS FIELDS

In this section, we employ the constant element scheme to discretize the CVBIE of Eq. (27) in the following

$$\frac{\alpha}{2\pi}\{\mathbf{f}\}=\left[\tilde{\mathbf{T}}\right]\{\mathbf{f}\}+[\mathbf{U}]\{\mathbf{g}\}, \tag{31}$$

where

$$\{\mathbf{f}\}=\left\{f_j\right\}_{N\times 1}, \{\mathbf{g}\}=\left\{g_j\right\}_{N\times 1}, \tag{32}$$

$$\left[\tilde{\mathbf{T}}\right]=\left[\tilde{T}_{jl}\right]_{N\times N}, [\mathbf{U}]=\left[U_{jl}\right]_{N\times N}, \tag{33}$$

in which, N is the number of the element and each element of influence matrices $\left[\tilde{T}_{jl}\right]$ and $\left[U_{jl}\right]$ is determined by

$$\tilde{T}_{jl}=\begin{cases}\dfrac{1}{2\pi i}C.P.V.\int_{\partial\Omega_l}\dfrac{1}{s_l-z_j}ds_l=0, & j=l,\\ \dfrac{1}{2\pi i}\int_{\partial\Omega_l}\dfrac{1}{s_l-z_j}ds_l=\dfrac{1}{2\pi i}\ln(s_l-z_j)\Big|_{s_l=s_l^L}^{s_l=s_l^R}, & j\neq l,\end{cases} \tag{34}$$

$$U_{jl}=\frac{1}{2\pi i}\int_{\partial\Omega_l}\ln\left|s_l-z_j\right|^2 ds_l=\frac{e^{-i\theta_l}}{\pi i}\int_{\partial\Omega_l}\ln\left|s_l-z_j\right|dt(s_l), \tag{35}$$

where s_l^L and s_l^R are the coordinates of the starting and ending points for the lth element, respectively, θ_l and $dt(s_l)$ are the argument and the path integral of the lth element, respectively, and

$$g_l = \left.\frac{\partial f(s)}{\partial s}\right|_{s=s_l}. \tag{36}$$

Since the collocation point is the central point of the corresponding element, the solid angle α is equal to π. After arrangement for Eq. (31), we have

$$[\mathbf{T}]\{\mathbf{f}\}+[\mathbf{U}]\{\mathbf{g}\}=\{\mathbf{0}\}, \tag{37}$$

where the influence matrix $[\mathbf{T}]$ is equal to

$$[\mathbf{T}]=[\tilde{\mathbf{T}}]-\frac{1}{2}[\mathbf{I}], \tag{38}$$

in which, $[\mathbf{I}]$ is the identity matrix. After substituting the boundary condition in Eq. (17) to Eq. (37), we have

$$[\mathbf{T}]\{\mathbf{f}\}+G\alpha\{\mathbf{p}\}=\{\mathbf{0}\}, \tag{39}$$

where $\{\mathbf{p}\}$ is equal to

$$\{\mathbf{p}\}=[\mathbf{U}]\{\mathbf{1}\}. \tag{40}$$

It is noted that there are $N+1$ unknown coefficients in the present approach for solving the stress fields of the pure torsion problem. To satisfy the traction free boundary condition, we can assume

$$f(z)=\bar{n}(z)\beta(z),\ z\in\partial\Omega, \tag{41}$$

where $\bar{n}(z)=n_x-in_y$ and $\beta(z)$ is the magnitude of $f(z)$ for the boundary point. By this way, two gains are obtained. Automatically satisfying the traction free boundary condition is the first one. The second one is that all unknown coefficients are real-valued. Also using the constant element scheme to discretize Eq. (41), we have

$$\{\mathbf{f}\}=[\bar{\mathbf{n}}]\{\boldsymbol{\beta}\}, \tag{42}$$

where

$$\{\boldsymbol{\beta}\}=\{\beta_l\}_{N\times 1}, \tag{43}$$

$$[\bar{\mathbf{n}}]=[\bar{n}_{jl}]_{N\times N}, \tag{44}$$

in which, $[\bar{n}_{jl}]$ is a diagonal matrix as given below:

$$\bar{n}_{jl}=\begin{cases}\bar{n}(z_j), & j=l,\\ 0, & j\neq l.\end{cases} \tag{45}$$

Substituting Eq. (42) to Eq. (39), we have

$$[\mathbf{T}\bar{\mathbf{n}}]\{\boldsymbol{\beta}\} + G\alpha\{\mathbf{p}\} = \{\mathbf{0}\}, \tag{46}$$

where

$$[\mathbf{T}\bar{\mathbf{n}}] = [\mathbf{T}][\bar{\mathbf{n}}]. \tag{47}$$

Furthermore, to easily calculate the static equivalence condition, we transform it into the form of contour integrals as shown below:

$$M = -G\alpha\int_{\partial\Omega}\left(\frac{x^3}{3}n_x + \frac{y^3}{3}n_y\right)dt(z) + \frac{1}{2}\int_{\partial\Omega} z\bar{z}\beta(z)dt(z). \tag{48}$$

Also, we employ the constant element scheme to discretize the static equivalence condition in Eq. (48) and we have

$$\{\mathbf{q}\}^T\{\boldsymbol{\beta}\} - GI_p\alpha = M, \tag{49}$$

where

$$\{\mathbf{q}\}^T = \{q_j\}^T_{1\times N}, \tag{50}$$

$$I_p = \int_{\partial\Omega}\left(\frac{x^3}{3}n_x + \frac{y^3}{3}n_y\right)dt(z), \tag{51}$$

in which,

$$q_j = \frac{1}{2}\int_{\partial\Omega_j} z\bar{z}\,dt(z_j). \tag{52}$$

Then, we have the following linear algebraic equation by combining Eq. (46) with Eq. (49)

$$\begin{bmatrix} [\mathbf{T}\bar{\mathbf{n}}] & \{\mathbf{p}\} \\ \{\mathbf{q}\}^T & -GI_p \end{bmatrix}_{(N+1)\times(N+1)} \begin{Bmatrix} \{\boldsymbol{\beta}\} \\ \alpha \end{Bmatrix}_{(N+1)\times 1} = \begin{Bmatrix} \mathbf{0} \\ M \end{Bmatrix}_{(N+1)\times 1}. \tag{53}$$

To ensure the influence matrix in Eq. (53) to be full rank, we update it as expressed below:

$$\begin{bmatrix} \mathrm{Re}([\mathbf{T}\bar{\mathbf{n}}]) & G\,\mathrm{Re}(\{\mathbf{p}\}) \\ \mathrm{Im}([\mathbf{T}\bar{\mathbf{n}}]) & G\,\mathrm{Im}(\{\mathbf{p}\}) \\ \{\mathbf{q}\}^T & -GI_p \end{bmatrix}_{(2N+1)\times(N+1)} \begin{Bmatrix} \{\boldsymbol{\beta}\} \\ \alpha \end{Bmatrix}_{(N+1)\times 1} = \begin{Bmatrix} \{\mathbf{0}\} \\ \{\mathbf{0}\} \\ M \end{Bmatrix}_{(2N+1)\times 1}. \tag{54}$$

It is noted that all elements of the influence matrix in Eq. (54) are real-valued. Since the influence matrice is over-determined, we employed the pseudoinverse matrix method to evaluate its inverse matrix. Regarding the present approach for solving the torsion problems, the torsional rigidity, D, can be obtained straightforward as expressed below:

$$D = \frac{M}{\alpha}. \tag{55}$$

Without loss of generality, we set the torque M to be 1 in the real implementation.

5 NUMERICAL EXAMPLES AND DISCUSSIONS

In this paper, we consider six cases to demonstrate the validity of the present complex variable boundary element method (CVBEM). The sketches of cross section are depicted in Figs. 2. For the circular case with a radius ($a = 2$) as shown in Fig. 2, numerical results for the torsional rigidity versus number of elements are listed in Table 1. Also, the convergence curve is plotted in Fig. 3. However, the rate of convergence is not fast. After comparing with the analytical solution, the numerical results obtained by the present CVBEM are acceptable. In the conventional boundary integral formulations, only the normal derivative is required and the tangent derivative is not. While the differential term in the present CVBIE contains both normal and tangent derivatives along the boundary. In our real implementation, the constant element scheme is employed. The rate of convergence is not fast, due to more constant element to simulate the tangent derivative. This can explain why lower number of element can not yield good result. Nevertheless, the present CVBEM still has its benefits for solving torsion problems. Not only the torsional rigidity but also the stress fields can be obtained straightforward. The stress flow of a circular bar subjected to a torque is plotted in Fig. 4.

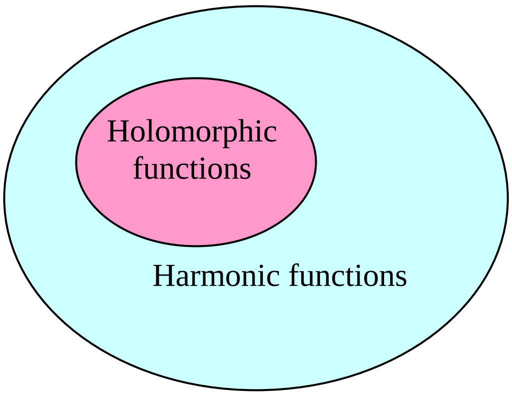

Figure 1: The sketch of relation between the set of holomorphic functions and harmonic functions

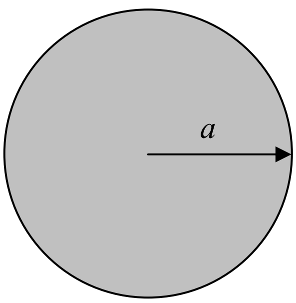

Figure 2: Acircular cross-section

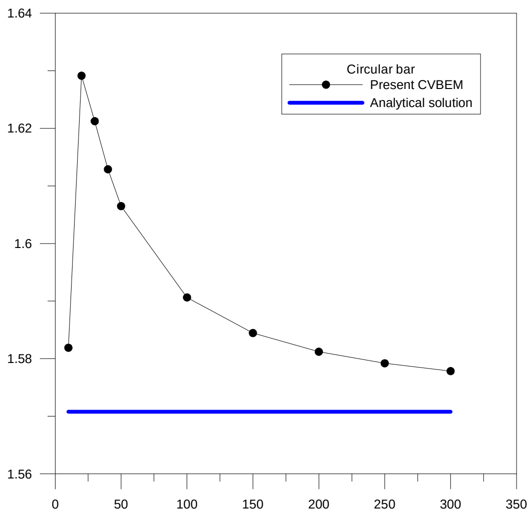

Figure 3: Torsional rigidity versus the number of elements

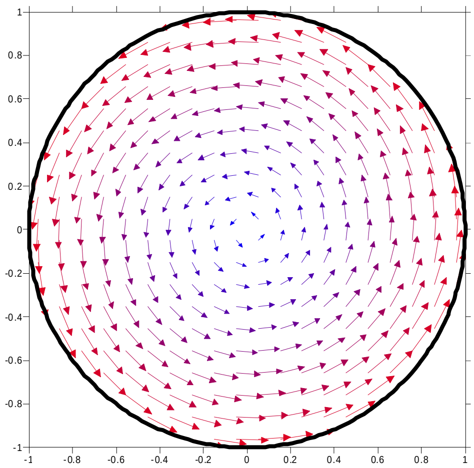

Figure 4: Vector field of the two shear stress fields

Table 1: Torsional rigidity of a circular bar

$a = 1$	Prensent CVBEM	Analytical solution	Relative error (%)
N=10	1.58189	1.57080	0.706
N=20	1.62912		3.713
N=30	1.62123		3.211
N=40	1.61288		2.679
N=50	1.60649		2.272
N=100	1.59063		1.263
N=150	1.58445		0.869
N=200	1.58120		0.662
N=250	1.57920		0.535
N=300	1.57784		0.448

6 CONCLUSIONS

- In this paper, we have successfully proposed a new CVBEM to solve the Saint-Venant's torsion problems.
- The present CVBEM can be derived from the Borel-Pompeiu formula in stress variables.
- Deferent from the conventional CVBEM based on the Cauchy-Riemann equations, the present approach not only can solve holomorphic (analytic) functions but also can solve harmonic functions.
- The main character of the present method is that we can directly solve the two shear stress fields at the same time.
- By using the present approach for solving the Saint-Venant's torsion problems, two benefits can be gained.
- The stress fields and the torsional rigidity can be determined straightforward without any numerical differentiation and integration again, respectively.
- From above view point, the present CVBEM is more general and convenient than the conventional one when solving the Saint-Venant's torsion problems, although the rate of convergence is not fast.

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