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TIKHONOV REGULARIZATION FOR THE MODIFIED MAPPING-COLLOCATION METHOD APPLIED TO CIRCUMFERENTIAL CRACK IN A CURVED BEAM

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Abstract. The modified mapping-collocation (MMC) method is applied to the problem of an isotropic curved beam with a circumferential crack under pure bending moment. Using least squares method to solve the overdetermined system of boundary condition equations causes numerical difficulties. Alternatively zeroth-order Tikhonov regularization is applied to solve the system. The regularization technique appears brilliantly helpful in eliminating the convergence problems associated with ill-posedness of the problem. The boundary condition satisfaction is examined. The results of stress analysis are in good agreement with that of the finite element method.

1 INTRODUCTION

Delamination (or interlaminar fracture) as a mechanism of failure in laminated composite materials is of high importance in aerospace and wind turbine industries. Generally occurring before other modes of damage, it appears as *the weakest link* (resistance to de-cohesion among laminae) and determines the ability of the part to bear load. The phenomenon is not only of high importance because of probable safety problems it may cause, but also in regard to effects of its consideration in design procedure on costs through body weight and material consumption. For thin-walled aircraft structures, stresses normal to the laminated panels are usually very low and therefore of no concern, however, for curved laminated parts, interlaminar tensile stresses become quite large. In this kind of structures the ability to analyze curved geometry of the crack come into prominence. Lu et al. used finite element analysis with ABAQUS to compute mode mixity and energy release rates for circumferential cracks under pure moment loading in the curved beams

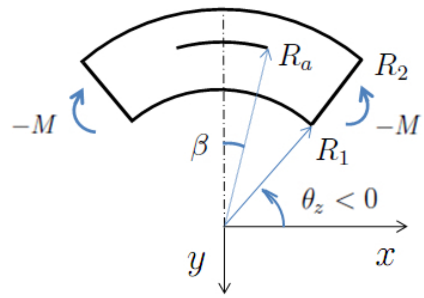

Figure 1: Schematic of the problem of a curved beam with a circumferential crack

with different othotropy degrees [1]. As a first step toward semi-analytic analysis of delamination in cylindrically orthotropic curved beams, an isotropic problem of the same geometry is considered.

Elastic analysis of a curved crack in an isotropic curved beam under pure bending (Figure 1) using the modified mapping-collocation (MMC) method [2] has been recently addressed by the authors [3]. The MMC method is a semi-analytic approach that combines complex analysis tools (mainly based on Muskhelishvili formulation [4]) with boundary collocation method. Muskhelishvili potential is expanded in a complex Laurent series form. The boundary condition equations are written for discrete points (stations) on the boundary. An overdetermined linear system of boundary condition equations is formed which its unknowns are coefficients of the series expansion. Therefore, accurate computation of the series coefficients is crucial in determining the stress field and thereafter stress intensity factors.

Solving the system resulted from the MMC method in a least square sense suffers some harsh restrictions on both geometrical parameters (such as beam thickness) and number of stations on the boundary, which makes doing convergence tests nearly impossible. However these numerical difficulties are originally associated with the known ill-posedness of the coefficient matrix obtained based on Laurent series expansion [5] of the Muskhelishvili potential, the difficulties are intensified in the case of curved cracks. Calculating stress intensity factor (SIF) for a curved crack using the MMC method leads to numerical instabilities, which are originally associated with the large exponents in the higher order terms in the truncated series expansion [5]. Lack of an analytical solution in general series form (like Westergaard solution for the straight crack) makes the solution for a curved crack more vulnerable numerically. In fact, the coefficient matrix of an overdetermined system which is obtained from the Laurent series expansion is a Vandermonde matrix (however it does not exactly match the present study) that is known to be severely ill-conditioned [6].

Approaching the ill-conditioned systems by classical methods (like least square min-

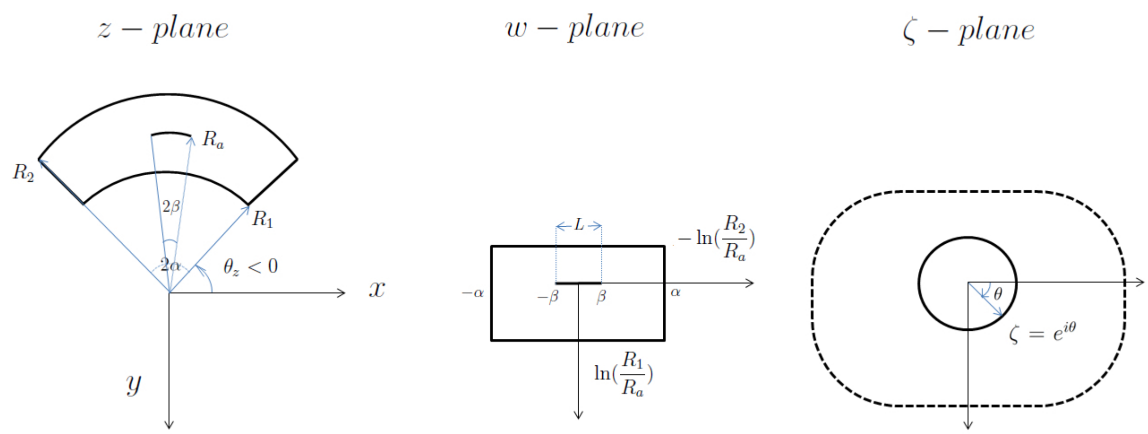

Figure 2: The successive mapping plan

imization) might fail or lead to the solutions that are either inexact or have very large norms. Tikhonov regularization is an effective method to find a reasonable solution for these kind of problems. In fact the method changes the classical least square problem to a minimization problem and then reduces that by using KKT (Karush–Kuhn–Tucker) conditions of optimality to a square system of equations that is *regularized* and ready to be solved normally, based on Gauss elimination [7].

2 THE MODIFIED MAPPING-COLLOCATION (MMC) METHOD

The modified mapping-collocation (MMC) method is described comprehensively in the recent study by the authors [3] and is briefly discussed here. According to G. V. Kolosov's formulation [4], only two complex analytic functions (e.g. $\phi(z)$ and $\psi(z)$) are needed in order to describe the stress field of a two-dimensional elastic body:

$$\sigma_y + \sigma_x = 4\Re\{\phi'(z)\} \tag{1}$$

$$\sigma_y - \sigma_x + 2i\sigma_{xy} = 2[\bar{z}\phi''(z) + \psi'(z)] \tag{2}$$

In order to map the unit circle in the image plane (ζ-plane) to a circumferential crack in the physical plane (z-plane), two successive mappings (figure 2) are combined to a single composite function $z = h(\zeta)$:

$$z = f(g(\zeta)) = h(\zeta) = R_a \exp\{i[\frac{\beta}{2}(\zeta + \zeta^{-1}) - \frac{\pi}{2}]\} \tag{3}$$

Using continuation argument of Kartzivadze in the image plane results in:

$$\psi(\zeta) = -\overline{\phi}(\frac{1}{\zeta}) - \frac{\overline{h}(\frac{1}{\zeta})}{h'(\zeta)}\phi'(\zeta), \qquad |\zeta| > 1 \tag{4}$$

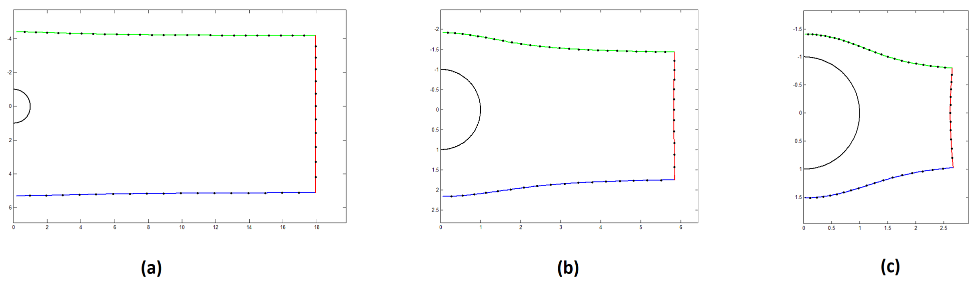

Figure 3: The mapped boundaries in the image plane for the beam with $R_1 = 24\ mm$, $R_2 = 36\ mm$ and $R_a = 30\ mm$. case **(a)**: $\beta = 5°$, case **(b)**: $\beta = 15°$ and case **(c)**: $\beta = 30°$. The unit circle (the mapped crack) is plotted in black. Due to symmetry only right half of the mapped region is presented.

which is of extreme importance from two aspects; it reduces number of *Muskhelishvili potentials* required to describe the stress field to one (i.e. $\phi(\zeta)$ only), and secondly it leads to automatic satisfaction of traction-free conditions on the crack boundary.

In the present solutions, only the local force boundary conditions are used to form the overdetermined system of boundary condition equations. Transferring Muskhelishvili formulation for local force boundary condition into the image plane, on each AB segment of the boundary:

$$b.i[\phi(\zeta) - \phi(\frac{1}{\bar{\zeta}}) + (h(\zeta) - h(\frac{1}{\bar{\zeta}}))\frac{\overline{\phi'(\zeta)}}{\overline{h'(\zeta)}}]_{\zeta_A}^{\zeta_B} = (F_x + iF_y)_{on\ \overline{AB}} \tag{5}$$

where b is beam depth in the direction perpendicular to the plane (here $b = 1\ m$). The right hand side expression is obtained from the classical solution of Golovin for crack-less curved beam [3].

Finally, the stress symmetry with respect to the imaginary axis leads to Laurent series expansion of the form:

$$\phi(\zeta) = \sum_{n=-M}^{N} [iA_{2n}\zeta^{2n} + A_{2n+1}\zeta^{2n+1}] \tag{6}$$

for Muskhelishvili potential, where A_{2n} and A_{2n+1} are purely real coefficients and M and N are non-negative integers. Thus the number of unknowns is $2(M + N) + 1$. The coefficient A_0 is eliminated since it is concerned with rigid body motion.

Substituting all of the corresponding terms and their derivatives into the local force boundary condition (equation (5)), for each ζ station on the boundary, two equations can be obtained (x and y direction force components). The number of equations is two times the number of stations. The ratio of the number of equations to the number of unknowns is known as *redundancy factor*. In the present study, the stations are chosen in equal distance from each other in the z-plane. The mapped region and stations in ζ-plane are plotted for a sample beam. Three different crack half angle, β, values are considered (figure 3). By solving the system for the unknown coefficients of the Muskhelishvili potential series expansion, stress components at each point of the body can be calculated by the equations (1) and (2) in the image plane [3].

3 TIKHONOV REGULARIZATION

The problem with an ill-conditioned system of equations is that minimization of the residual of the system (as in least square method) do not necessarily lead to a close approximation to the true solution. A mathematical description of this expression is given by the relation below:

$$\|\hat{\mathbf{x}} - \mathbf{x}\| \leq \|\mathbf{A}^{-1}\| \; \|\mathbf{r}(\hat{\mathbf{x}})\| \tag{7}$$

When $\|\mathbf{A}^{-1}\|$ is large, it is quite possible to have a *bad* approximation in spite of having a small residual [8].

Applying the MMC method to the problem of a circumferential crack in a curved beam, it was observed [3] that using least square method to solve the overdetermined system of local force boundary condition equations causes some numerical difficulties. Numerical instability was increasing by increase in the number of stations on the boundary. However the residual values on the boundary was small, the solution was behaving unstably and was sensitive to the geometrical parameters.

Tikhonov regularization is a numerical procedure which aims at stabilizing the systems suffering ill-posedness. For a system with a rank-deficient or very nearly singular coefficient matrix, least squares method (which manages to minimize $\Phi(\mathbf{x}) = \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2$) often gives solutions that vary unstably. To stabilize the computation, a penalty term is added in order to reduce the large components, thus the target is altered to minimization of [8]:

$$\Phi_\alpha(\mathbf{x}) = \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2 + \alpha\|\mathbf{x}\|_2^2 \tag{8}$$

where α is called regularization parameter. It can be shown that using KKT conditions the minimization problem above can be reduced to:

$$(\mathbf{A}^T\mathbf{A} + \alpha\mathbf{I})\mathbf{x}_\alpha = \mathbf{A}^T\mathbf{b} \tag{9}$$

which is called zeroth order Tikhonov regularization or SVD (singular value decomposition) with damping. The regularization parameter, α, is found by tabulating the residuals of the *regularized* system (9) for α values of different order of magnitude, which is called as *Morozov discrepancy principle* [8].

4 RESULTS

The MMC method solves a BVP of 2-D elasticity and computes the stress field. Boundary condition satisfaction is monitored to ensure accuracy of the solution. Since the

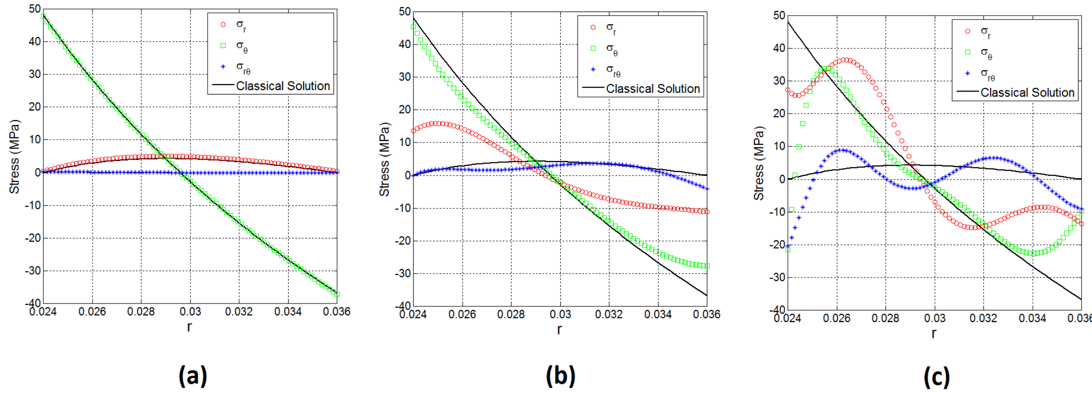

Figure 4: Instability of the solution *without* applying Tikhonov regularization method: The stress components on the moment-exerted boundary of the curved beam with $R_1 = 24\ mm$, $R_2 = 36\ mm$, $R_a = 30\ mm$ and $\beta = 5°$. Case **(a)**: Total number of 14 stations, $M = N = 3$, Case **(b)**: Total number of 24 stations, $M = N = 5$, Case **(c)**: Total number of 33 stations, $M = N = 8$.

boundary conditions on the moment-exerted boundary of the curved beam are specified according to the classical (crack-less) solution, it is expected that the same values are obtained from the solution of the BVP. The results are computed for a *sample beam* with inner radius, R_1, of 24 mm, outer radius, R_2, of 36 mm and crack radius, R_a, of 30 mm. For the sample beam with $\beta = 5°$, the stress components computed by the MMC method *with no regularization* are shown in figure 4. It can be seen that the boundary condition satisfaction decays swiftly as the number of stations (and proportionally the number of series terms) increase.

The values of the stress intensity factors (SIF) do also vary unstably as the size of the system increases. The SIF values are directly calculated from the stress values in front of the crack tip using the stress correlation method [9]. Surprisingly, for the *regularized* solutions, a very stable trend is observed. Even with a station number of 300 and $M = N = 38$ (for $\alpha = 10^{-8}$) the solution satisfies the boundary conditions almost exactly. Another point to mention is rapid convergence of the SIF values after the application of the Tikhonov regularization. The convergence was checked for different system redundancy factors $2, 3$ and 4.

For the *sample beam* the non-dimensionalized energy release rate values computed by Tikhonov-regulated MMC method are compared with FEM (ABAQUS) results given in [1] for six different half arc angle values, namely $\beta = 5°, 10°, 15°, 20°, 25°$ and $30°$ (figure 5-(a)). The non-dimensionalized parameter is:

$$\frac{GEH^3}{M^2} = \frac{(K_I^2 + K_{II}^2)(R_2 - R_1)^3}{M^2} \tag{10}$$

where M is moment per unit depth of the beam. Also the mode mixity values are plotted against β (figure 5-(b)).

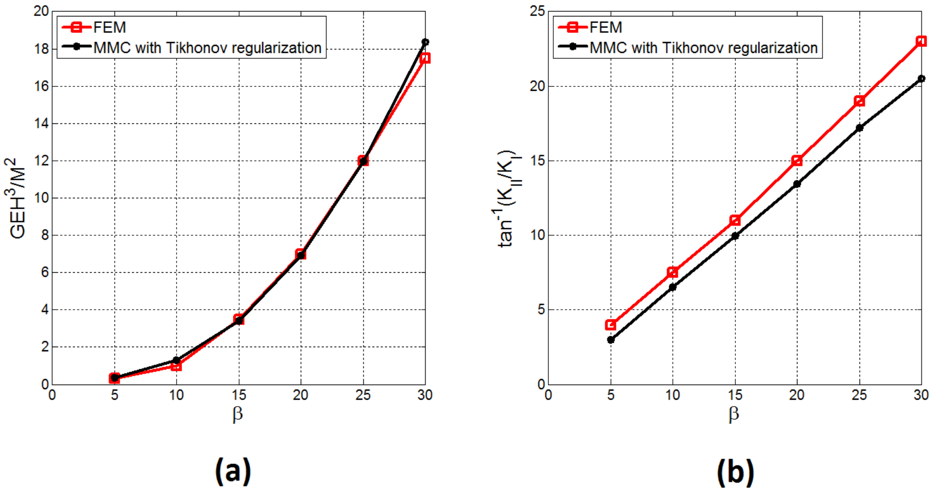

Figure 5: The results of Tikhonov-regularized MMC method (in black) and FEA with ABAQUS (in red, according to Lu et al. [1]) for **(a)** the nondimentionalized energy release rate and **(b)** mode mixity plotted against crack half arc angle, β, (in degrees) for the *sample beam*.

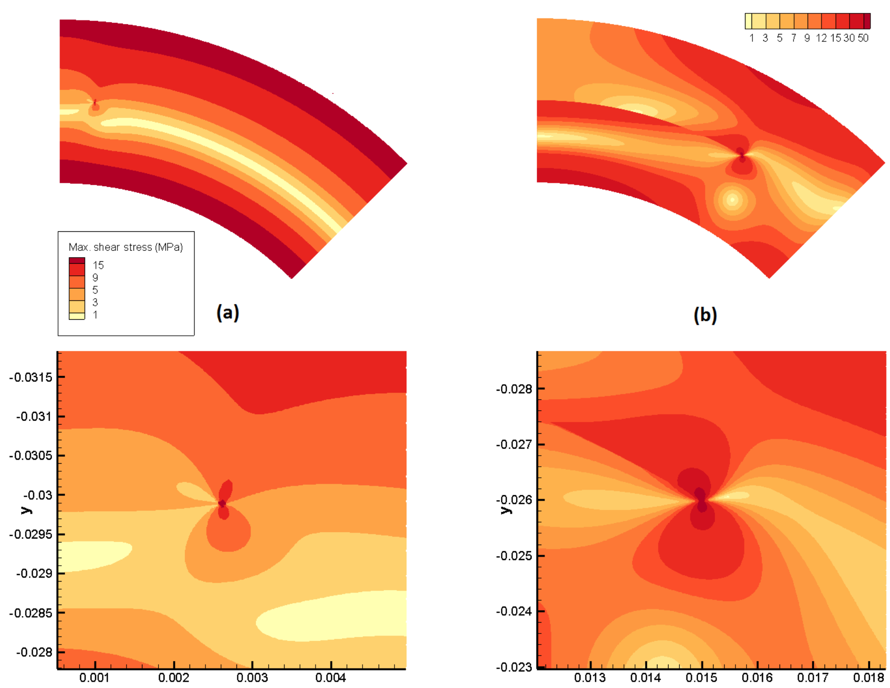

Figure 6: Contours of constant τ_{max} with the corresponding zoomed view of the crack tip obtained by Tikhonov-regularized MMC method for **(a)** $\beta = 5°$ and **(b)** $\beta = 30°$, for the *sample beam*.

After computing the coefficients of Muskhelishvili potential expansion, stress components at each point of the body can be obtained from Kolosov's complex stress field representation in the image plane [3]. The contours of constant τ_{max} are shown for two different crack half arc angles in the sample beam case (figure 6). The depth of the beam (in x_3 direction) is assumed as 1 m.

5 CONCLUSIONS

Tikhonov regularization is successfully utilized in the numerical part of the modified mapping collocation (MMC) method in order to solve the problem of a circumferential crack in an isotropic curved beam under pure bending. The target of analysis is to obtain accurate values of stress intensity factors and thereafter energy release rate. The results are compared with the values obtained by the finite element method [1] for a sample case and it is shown that there is a good agreement in terms of either SIF values or mode mixity.

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