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# NUMERICAL ALGORITHMS FOR PLASTICITY MODELS WITH NONLINEAR KINEMATIC HARDENING

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**Abstract.** In the present article numerical algorithms are adopted for the computational modeling of plasticity problems with nonlinear kinematic hardening rules. A solution procedure which preserves a quadratic rate of asymptotic convergence is used for the simulation of plasticity with nonlinear kinematic hardening. Computational applications and numerical results are illustrated. A comparative analysis is made between the adoption of the assumptions of linear versus nonlinear kinematic hardening rule by considering different types of material properties.

## 1 INTRODUCTION

Computational modeling of rate independent plasticity has nowadays achieved significant improvements, both in the proper formulation of the theoretical laws governing the inelastic behavior and in the definition of the numerical procedures for the integration of the boundary value problem, see among others Simo and Hughes [1] and Zienkiewicz Taylor and Fox [2]. In the numerical integration of rate independent plasticity problems the assumption of a linear kinematic hardening behavior is often adopted. This assumption proves to be advantageous from the numerical point of view since it often provides numerical algorithms characterized by computational efficiency and a symmetric tangential stiffness matrix. Nevertheless, in the last decades in the literature it has been outlined the necessity of adopting nonlinear kinematic hardening rules in order to properly model experiments on real materials, see e.g. Armstrong and Frederick [3], Dafalias and Popov [4], McDowell [5], Chaboche [6], Lubliner et al. [7]. This is especially true under cyclic loading conditions and when simulating experiments on solid materials subject to loading unloading and reloading processes, see for instance Chaboche [8] and Auricchio and Taylor [9]. For a review paper see e.g. Chaboche [10].

However, the adoption of nonlinear kinematic hardening rules for plasticity models is not a trivial task to be accomplished in the computational procedures. In fact it poses a number of challenges if a fast and robust computational solution is to be pursued. As a matter of fact the investigation of fast and robust integration methods for nonlinear kinematic hardening models and complex loading conditions currently represents an active topic of research. At this regard see for instance Auricchio and Taylor [9], Chaboche and Cailletaud [11], Hartmann et al. [12], Dettmer and Reese [13], Nukala [14], Artioli et al. [15]. In the present article an integration scheme is applied which preserves a quadratic rate of asymptotic convergence in the simulation of plasticity models with nonlinear kinematic hardening laws. A comparative analysis is made between the linear and the nonlinear kinematic hardening assumption by selecting different types of material parameters. Some considerations are presented regarding the applicability of linear and nonlinear kinematic hardening rules in the computational simulation of material behavior. Finally, computational applications and numerical results are reported in order to illustrate the effectiveness of the algorithmic procedure.

## 2 THE CONTINUUM EVOLUTIVE MODEL

We consider the body $\mathcal{B}$ in the reference configuration $\Omega \subset \Re^n$, with $1 \leq n \leq 3$. Let us indicate with $\mathcal{T} \subset \Re_+$ the time interval of interest and with $\boldsymbol{V}$ the space of displacements, $\boldsymbol{D}$ the strain space and $\boldsymbol{S}$ the dual stress space. We also denote by $\mathbf{u} : \Omega \times \mathcal{T} \rightarrow \boldsymbol{V}$ the displacement and by $\boldsymbol{\sigma} : \Omega \times \mathcal{T} \rightarrow \boldsymbol{S}$ the stress tensor. The compatible strain tensor is defined by $\boldsymbol{\varepsilon} = \nabla^s \mathbf{u} : \Omega \times \mathcal{T} \rightarrow \boldsymbol{D}$, where $\nabla^s$ is the symmetric part of the gradient.

The stress tensor is assumed to be additively decomposed into the deviatoric and spherical parts

$$\boldsymbol{\sigma} = \mathbf{s} + p\,\mathbf{1}, \tag{1}$$

where $\mathbf{s} = \text{dev}\boldsymbol{\sigma} = \boldsymbol{\sigma} - p\,\mathbf{1}$ is the stress deviator, $p = \frac{1}{3}tr(\boldsymbol{\sigma})$ is the pressure of the spherical part $p\,\mathbf{1}$ and $\mathbf{1}$ is the rank two identity tensor. The strain tensor is accordingly decomposed into the deviatoric and volumetric parts

$$\boldsymbol{\varepsilon} = \mathbf{e} + \frac{1}{3}\,\theta\,\mathbf{1}, \tag{2}$$

where $\mathbf{e} = \text{dev}\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon} - \frac{1}{3}\,\theta\,\mathbf{1}$ is the strain deviator and $\theta$ is the change in volume.

The elastic relation between the volumetric part of the stress and the volumetric part of the strain is expressed by

$$p = K\,\theta, \tag{3}$$

where $K$ is the bulk modulus.

The linear elastic relation between the deviatoric stress and the elastic deviatoric strain is given by

$$\mathbf{s} = 2G\,\mathbf{e}^e = 2G[\mathbf{e} - \mathbf{e}^p], \tag{4}$$

where $G$ is the shear modulus and the deviatoric strain is additively decomposed into the elastic and the plastic part, according to $\mathbf{e} = \mathbf{e}^e + \mathbf{e}^p$.

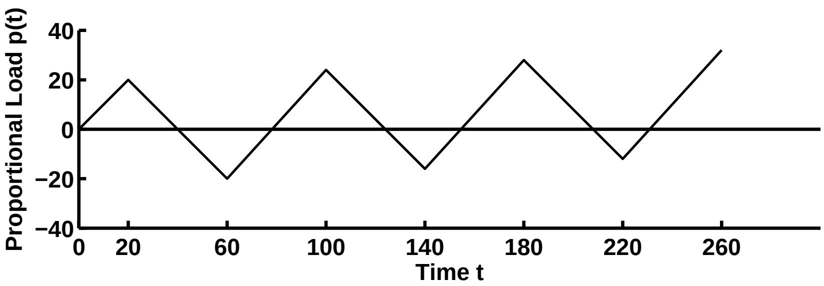


**Figure 1**: Cyclic loading program in tension-compression with increasing mean stress.

The relative stress $\boldsymbol{\Sigma}$ is expressed by

$$\boldsymbol{\Sigma} = \mathbf{s} - \boldsymbol{\alpha}, \tag{5}$$

where $\boldsymbol{\alpha}$ represents the deviatoric back stress.

We consider the equations governing the behavior of a $J_2$ material model. Consequently, a von Mises yield criterion is adopted in the form

$$f(\boldsymbol{\sigma}, \boldsymbol{\alpha}, \kappa) = \|\mathrm{dev}\boldsymbol{\sigma} - \boldsymbol{\alpha}\| - \kappa(\chi_{iso}) = \|\mathbf{s} - \boldsymbol{\alpha}\| - \sqrt{\frac{2}{3}}(\sigma_{yo} + \chi_{iso}) \leq 0, \tag{6}$$

where $\kappa(\chi_{iso}) = \sqrt{\frac{2}{3}}\,\sigma_y = \sqrt{\frac{2}{3}}(\sigma_{yo} + \chi_{iso})$ represents the current radius of the yield surface in the deviatoric plane and $\sigma_{yo}$ denotes the uniaxial yield stress of the virgin material. For a linear isotropic hardening behavior the static internal variable related to isotropic hardening is specified by $\chi_{iso} = H_{iso}\overline{e}^p$, where the dual kinematic internal variable $\overline{e}^p$ represents the equivalent (accumulated) plastic strain $\overline{e}^p = \int_0^t \sqrt{\frac{2}{3}}\|\dot{\mathbf{e}}^p\|\mathrm{d}\,t$.

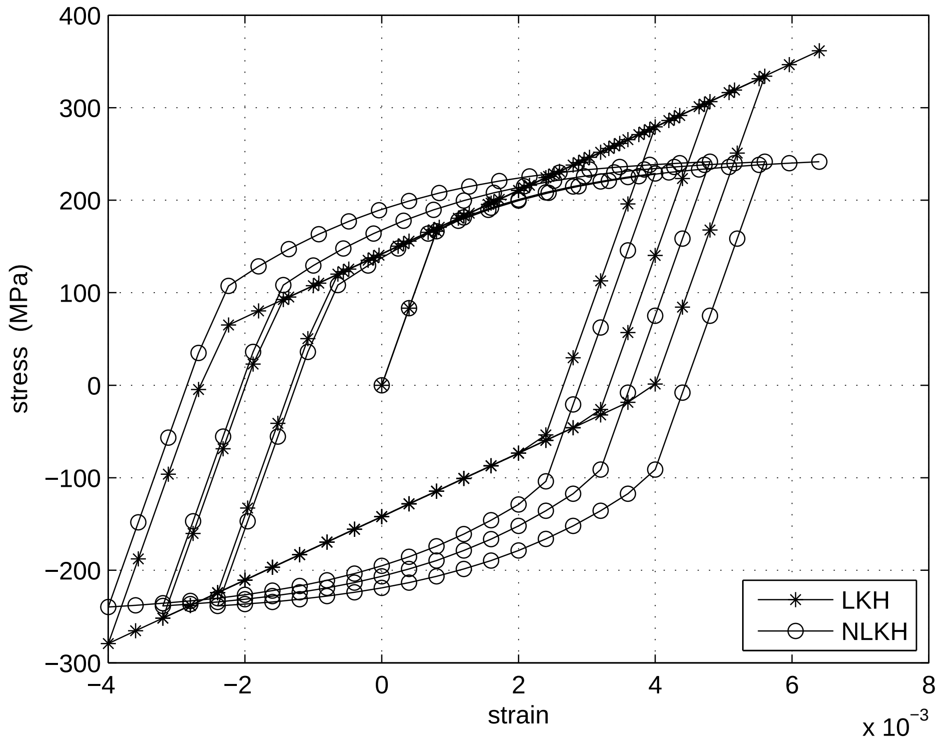


**Figure 2**: Cyclic loading condition in tension-compression for the steel X5CrNi 18·9 (loading program given in fig.1). Comparison between linear kinematic hardening (LKH) and nonlinear kinematic hardening (NLKH) in plasticity.

For nonlinear isotropic hardening it is often assumed

$$\chi_{iso} = H_{iso}(\bar{e}^p)^m, \qquad \text{or} \qquad R = R_\infty(1 - \mathrm{e}^{-b\,\bar{e}^p}), \tag{7}$$

where $m$, $R_\infty$ and $b$ are material parameters.

For associative plasticity the evolutive flow law for the deviatoric plastic strain is expressed by

$$\dot{\mathbf{e}}^p = \dot{\gamma}\,\frac{\partial f}{\partial \boldsymbol{\sigma}} = \dot{\gamma}\,\frac{\partial f}{\partial \boldsymbol{\Sigma}} = \dot{\gamma}\,\mathbf{n}, \tag{8}$$

where $\dot{\gamma}$ is the plastic multiplier and the second rank tensor $\mathbf{n}$ is defined by

$$\mathbf{n} = \frac{\boldsymbol{\Sigma}}{\|\boldsymbol{\Sigma}\|} \tag{9}$$

and it has unit norm.

The equivalent plastic strain rate is accordingly given by

$$\dot{\bar{e}}^p = \sqrt{\frac{2}{3}}\,\dot{\gamma}. \tag{10}$$

The back stress rate is expressed in the assumption of linear kinematic hardening behavior by

$$\dot{\boldsymbol{\alpha}} = \frac{2}{3}\,H_{kin}\,\dot{\mathbf{e}}^p. \tag{11}$$

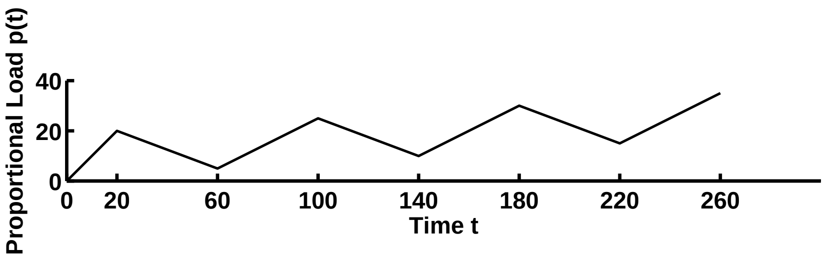


**Figure 3**: Cyclic loading program with increasing levels of loading.

In the literature for the nonlinear kinematic hardening behavior it is often adopted the relation originally proposed by Armstrong and Frederick [3]

$$\dot{\boldsymbol{\alpha}} = \frac{2}{3}\,H_{kin}\,\dot{\mathbf{e}}^p - H_{nl}\,\dot{\bar{e}}^p\boldsymbol{\alpha}, \tag{12}$$

where $H_{nl}$ is a non-dimensional material dependent parameter characterizing nonlinear kinematic hardening behavior and $H_{nl} = 0$ stands for linear kinematic hardening.

A better approximation of the nonlinear kinematic hardening behavior results in adding several components of the back stress, with different recall constants, see e.g. Chaboche [10],

$$\boldsymbol{\alpha} = \sum_{i=1}^{M}\boldsymbol{\alpha}_i, \qquad \dot{\boldsymbol{\alpha}}_i = \tfrac{2}{3}\,H_{kin,i}\,\dot{\mathbf{e}}^p - H_{nl,i}\,\dot{\bar{e}}^p\,\boldsymbol{\alpha}_i. \tag{13}$$

## 3 NUMERICAL RESULTS

The investigation for fast and robust integration methods relative to nonlinear kinematic hardening models is currently a topic of active research in the literature. In the present paper an efficient solution procedure is used which preserves the quadratic rate of asymptotic convergence typical of Newton's schemes. The procedure is illustrated in detail in De Angelis and Taylor [16] where the development of the associated consistent tangent operator is also addressed for nonlinear kinematic hardening plasticity.

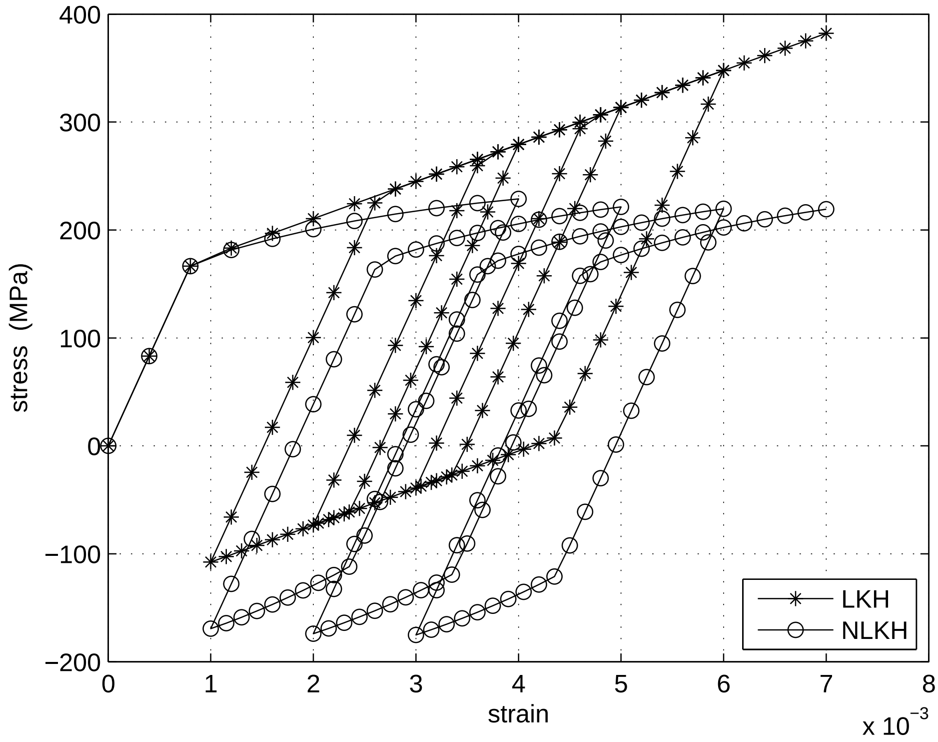


**Figure 4**: Cyclic loading condition with increasing levels of loading for the steel X5CrNi 18·9 (loading program given in fig.3). Comparison between linear kinematic hardening (LKH) and nonlinear kinematic hardening (NLKH) in plasticity.

In the sequel, by adopting the numerical integration procedure reported in detail in [16], a comparative analysis between the linear and the nonlinear kinematic hardening assumption is illustrated and discussed.

The numerical simulations are performed by using a three-dimensional finite element, based on a mixed approach (Simo et al. [17]) and implemented into the Finite Element Analysis Program (FEAP), see Zienkiewicz Taylor and Fox [2] and Taylor [18].

In the numerical tests a cubic specimen of side length set equal to 5 is loaded by imposing a uniform displacement at the top boundary of the specimen and with the appropriate boundary conditions. The sample is modeled with only one element. In the numerical examples different types of material properties have been assumed in order to analyze the effectiveness of the adoption of a linear versus a nonlinear kinematic hardening rule for the simulation of the material constitutive behavior.

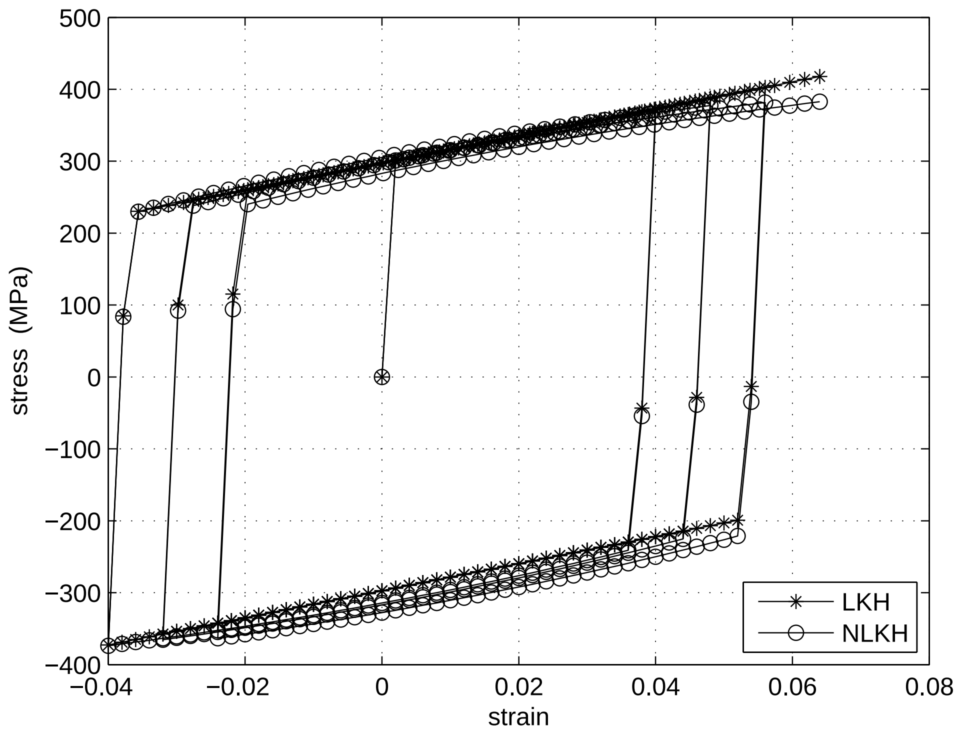


**Figure 5**: Cyclic loading conditions in tension-compression for the mild steel Ck 15 (loading program given in fig.1). Comparison between linear kinematic hardening (LKH) and nonlinear kinematic hardening (NLKH) in plasticity.

### 3.1 *Example 1*

We consider as a first example the material properties which identify the steel X5CrNi 18·9, see e.g. Hartmann et al. [12]. The material properties are: elastic modulus $E = 208000$ MPa, Poisson's ratio $\nu = 0.3$, yield limit $\sigma_{yo} = 170$ MPa, kinematic hardening modulus $H_{kin} = 41080$ MPa, nonlinear kinematic hardening parameter $H_{nl} = 525$, isotropic hardening modulus $H_{iso} = 0$ MPa. The prescribed displacement at the top boundary has been set equal to $u_o = 0.001$. The evolution with time of the proportional load coefficient $p(t)$ amplifies the prescribed displacement and describes the loading history according to relation $u(t) = p(t)u_o$.

A cyclic loading program in tension-compression with increasing mean stress has been performed by adopting the loading history reported in Figure 1. The related stress-strain curves are illustrated in Figure 2, which also shows the comparison of a linear kinematic hardening behavior (LKH) and a nonlinear kinematic hardening behavior (NLKH).

A cyclic loading history with increasing levels of loading is given in Figure 3. For this cyclic loading condition the stress-strain curves are illustrated in Figure 4 with the comparison of linear kinematic hardening rule (LKH) and nonlinear kinematic hardening rule (NLKH) in plasticity.

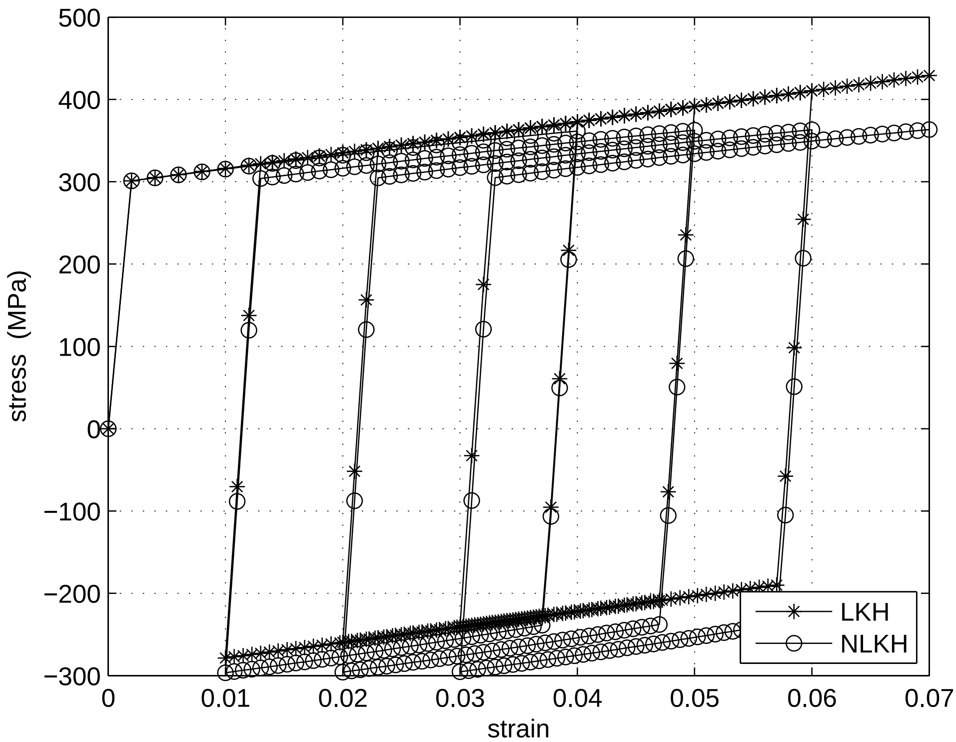


**Figure 6**: Cyclic loading conditions with increasing levels of loading for the mild steel Ck 15 (loading program given in fig.3). Comparison between linear kinematic hardening (LKH) and nonlinear kinematic hardening (NLKH) in plasticity.

### 3.2 *Example 2*

We consider as a second example the material properties which identify the mild steel Ck15, see Dettmer and Reese [13] and Lührs et al. [19]. The material properties are: elastic modulus $E = 208000$ MPa, Poisson's ratio $\nu = 0.3$, yield limit $\sigma_{yo} = 300$ MPa, kinematic hardening modulus $H_{kin} = 1900$ MPa, nonlinear kinematic hardening parameter $H_{nl} = 8.5$, isotropic hardening modulus $H_{iso} = 0$ MPa. The prescribed displacement at the top boundary has been set equal to $u_o = 0.01$. The evolution with time of the proportional load coefficient $p(t)$ describes the loading history $u(t) = p(t)u_o$.

The cyclic loading program in tension-compression given in Figure 1 has been adopted for the numerical simulation. The related stress-strain curves for the linear kinematic hardening behavior (LKH) and the nonlinear kinematic hardening behavior (NLKH) are reported in Figure 5.

Furtherly, the cyclic loading program with increasing levels of loading has been assigned as illustrated in Figure 3. For this cyclic loading condition the stress-strain curves are reported in Figure 6. For the assigned material properties Figure 6 also illustrates the comparison of a linear kinematic hardening rule (LKH) and a nonlinear kinematic hardening rule (NLKH) in plasticity.

## 4 CONCLUSIONS

In finite element applications linear kinematic hardening rules are frequently adopted in plasticity since this assumption ensures a symmetric tangent stiffness matrix and computationally efficient solution procedures. However, in the last years, in the literature it has been outlined the necessity of adopting nonlinear kinematic hardening rules in order to properly simulate experiments on real materials.

In the present article a numerical approach has been adopted for the computational simulation of nonlinear kinematic hardening rules in plasticity, see [16] for more details. The numerical approach is characterized by efficiency and effectiveness by preserving a quadratic rate of asymptotic convergence in the finite element solution procedure. Numerical examples and applications have been shown in order to illustrate the suitability of the approach to plasticity problems with nonlinear kinematic hardening rules. Computational applications and numerical results have been reported by applying the proposed procedure to different materials. A comparative analysis has been presented for the different material properties between the assumptions of linear versus nonlinear kinematic hardening laws. The performed analysis allows to have a better insight on the conditions upon which the adoption of the assumption of a linear versus a nonlinear kinematic hardening rule may be considered as effective.

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