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A HYSTERETIC MITC9 SHELL FINITE ELEMENT

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Abstract. In this work, a hysteretic shell finite element for the inelastic static and dynamic analysis of structures is presented. The Bouc-Wen model is utilized as a smooth hysteretic, rate independent model, capable of expressing the hysteretic behavior that can be easily extended to account for stiffness degradation, strength deterioration and pinching phenomena. On the basis of the classical theory of plasticity, the generalized 3D Bouc-Wen model is expressed in tensorial form incorporating the yield criterion and linear or nonlinear, isotropic or kinematic hardening law. Based on this approach, a hysteretic shell finite element is developed following the steps of FEM in which the shell is considered as a number of fully bonded layers. The classical elastic formulation of the shell element is employed following the Mixed Interpolation of Tensorial Components (MITC) approach in MITC9 shell element. This is extended by considering as additional hysteretic degrees of freedom the plastic strains at the Gauss points of each interface between layers, the evolution of which is determined by a Bouc- Wen evolution equation. The solution provides the nodal displacements, the elastic and plastic strains and the stresses at every Gauss point of each interface. Numerical results are presented that validate the proposed formulation, which are further compared against those obtained using Abaqus code. A good agreement is achieved between the standard FEM and the proposed formulation which computationally results as more efficient for the same accuracy.

1 INTRODUCTION

To avoid or relax locking phenomena in shell elements, which lead to mathematical instabilities, significant progress has been made on treating the mathematical inconsistencies and the overall error. The Mixed-Interpolated Tensorial Components - MITC elements constitute an efficient approach. In this work the MITC9 shell element is extended to the elastoplastic analysis of shell structures by incorporating the Bouc – Wen hysteresis model. Additional hysteretic degrees of freedom are introduced which control the plastic strains in a number of layers. At every Gauss point of each layer, the plastic strains are defined as additional hysteretic degrees of freedom, the evolution of which is described by Bouc- Wen evolution equations. In this way, the elastic and hysteretic stiffness matrices of shell elements are derived which subsequently are assembled to form the equations of motion. The evolution equations for the hysteretic degrees of freedom constitute an additional set of first order nonlinear equations which together with the linear equations of motion describe the inelastic problem. The equations of motion are solved following Newmark's scheme whereas the system of nonlinear evolution equations is established on the basis of a Livermore integration scheme. Numerical results are presented that justify the validity and accuracy of the proposed formulation.

2 INELASTIC BEHAVIOR AND BOUC – WEN HYSTERETIC MODEL

The response of most materials up to certain extend, can be considered elastic. In the elastic range these materials exhibit no internal damage, thus returning to zero stress – strain when unloaded. In stress space, the elastic domain is delimited by an external boundary i.e. the yield surface that is defined by a yield function of the form:

$$\Phi\left(\sigma_{ij},\sigma_y^0\right)=f\left(\sigma_{ij}\right)-\sigma_y^0=0 \tag{1}$$

where σ_y^0 is the initial yield stress, whereas any admissible stress state must satisfy the condition $\Phi\left(\sigma_{ij},\sigma_y^0\right)\le 0$.

Loading further the material, plastic yielding (or plastic flow), i.e. evolution of plastic strains is initiated, manifested as permanent strains at unloading. This is described by the plastic flow rule:

$$\underset{(\textit{tensor components notation})}{\dot{\varepsilon}_{ij}^{pl}=\dot{\lambda}\frac{\partial Q\left(\sigma_{ij}\right)}{\partial\sigma_{ij}}} \quad or \quad \underset{(\textit{tensor notation})}{\dot{\boldsymbol{\varepsilon}}^{pl}=\dot{\lambda}\frac{\partial Q(\boldsymbol{\sigma})}{\partial\boldsymbol{\sigma}}} \quad or \quad \underset{(\textit{matrix-vector notation})}{\left\{\dot{\varepsilon}^{pl}\right\}=\dot{\lambda}\frac{\partial Q\left(\{\sigma\}\right)}{\partial\{\sigma\}}} \tag{2}$$

where $Q\left(\sigma_{ij}\right)$ is a plastic potential function and $\dot{\lambda}$ is the plastic multiplier. For most civil engineering materials, excluding soils, a common valid approach is to associate the plastic potential with the yield function, $Q\left(\sigma_{ij}\right)=\Phi\left(\sigma_{ij}\right)$ (associated flow rule) and express eq. (2) as:

$$\dot{\varepsilon}_{ij}^{pl}=\dot{\lambda}\frac{\partial\Phi\left(\sigma_{ij}\right)}{\partial\sigma_{ij}} \quad or \quad \dot{\boldsymbol{\varepsilon}}^{pl}=\dot{\lambda}\frac{\partial\Phi(\boldsymbol{\sigma})}{\partial\boldsymbol{\sigma}} \tag{3}$$

A complementarity condition holds, i.e. $\dot{\lambda}\cdot\Phi=0$, that requires that either the yield function is zero, or $\dot{\lambda}$ is zero and there is no plastic flow. Together with the evolution of the plastic strain, an evolution of the yield stress itself is also manifested (hardening) and the yield surface undergoes expansion and/or translation.

In the case of kinematic hardening, which is capable of predicting the Bauschinger effect, the yield function is expressed in the form:

$$\Phi=\Phi\left(\sigma_{ij},\sigma_y^0\right)=f\left(\sigma_{ij}-\alpha_{ij}\right)-\sigma_y^0=0 \tag{4}$$

where $\boldsymbol{\alpha}$ is a tensorial back stress hardening parameter, that represents the evolution of the centre of the yield surface in the stress space. The back stress evolves as a function of the plastic multiplier, $\dot{\lambda}$ and the hardening function $\mathbf{G}$ as:

$$\underset{(tensor\ notation)}{\dot{\boldsymbol{\alpha}} = \dot{\lambda}\,\mathbf{G}} \qquad or \qquad \underset{(tensor\ components\ notation)}{\dot{\alpha}_{ij} = \dot{\lambda}\,G_{ij}} \qquad or \qquad \underset{(matrix-vector\ notation)}{\{\dot{a}\} = \dot{\lambda}\,\{G\}} \tag{5}$$

In the case of Prager's linear kinematic hardening, evolution of the back stress is defined by the following linear relation [6]:

$$\{\dot{\alpha}\} = C_p\left\{\dot{\varepsilon}^{pl}\right\} = C_p\left(\dot{\lambda}\frac{\partial\Phi}{\partial\{\sigma\}}\right) = \dot{\lambda}\{G\}, \qquad where \quad \{G\} = C_p\frac{\partial\Phi}{\partial\{\sigma\}} \tag{6}$$

where C_p is the hardening constant.

In addition the total strain tensor is considered as the sum of an elastic component ε_{ij}^{el} and a plastic component ε_{ij}^{pl} (assumption of additive decomposition) and thus:

$$\varepsilon_{ij} = \varepsilon_{ij}^{el} + \varepsilon_{ij}^{pl} \quad or \quad \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{el} + \boldsymbol{\varepsilon}^{pl} \tag{7}$$

Furthermore the stress increment is linearly related to the elastic strain increment in the plastic region and can be expressed by the following constitutive relation [2]:

$$\begin{gathered}\dot{\sigma}_{ij} = C_{ijkl}\,\dot{\varepsilon}_{kl}^{el} = C_{ijkl}\left(\dot{\varepsilon}_{kl} - \dot{\varepsilon}_{kl}^{pl}\right) \quad or \quad \dot{\boldsymbol{\sigma}} = \mathbf{C}:\dot{\boldsymbol{\varepsilon}}^{el} = \mathbf{C}:\left(\dot{\boldsymbol{\varepsilon}} - \dot{\boldsymbol{\varepsilon}}^{pl}\right) \quad or \\ \{\dot{\sigma}\} = [C]\left\{\dot{\varepsilon}^{el}\right\} = [C]\left(\{\dot{\varepsilon}\} - \left\{\dot{\varepsilon}^{pl}\right\}\right)\end{gathered} \tag{8}$$

For plastic flow to occur, the stresses must remain on the yield surface (consistency condition) and hence:

$$\dot{\Phi} = \left\{\frac{\partial\Phi}{\partial\{\sigma\}}\right\}^{\mathrm{T}} d\{\sigma\} + \left\{\frac{\partial\Phi}{\partial\{\alpha\}}\right\}^{\mathrm{T}} d\{\alpha\} = 0 \quad or \quad \dot{\Phi} = \frac{\partial\Phi}{\partial\boldsymbol{\sigma}}:d\boldsymbol{\sigma} + \frac{\partial\Phi}{\partial\boldsymbol{\alpha}}:d\boldsymbol{\alpha} = 0 \tag{9}$$

In order to find $\dot{\lambda}$, eq. (8) is pre-multiplied by the flow vector $\left\{\partial\Phi/\partial\{\sigma\}\right\}^{\mathrm{T}}$ and using eq. (9) and eq. (3):

$$\dot{\lambda} = \left(-\left\{\frac{\partial\Phi}{\partial\{\alpha\}}\right\}^{\mathrm{T}}\{G\} + \left\{\frac{\partial\Phi}{\partial\{\sigma\}}\right\}^{\mathrm{T}}[C]\left\{\frac{\partial\Phi}{\partial\{\sigma\}}\right\}\right)^{-1}\left\{\frac{\partial\Phi}{\partial\{\sigma\}}\right\}^{\mathrm{T}}[C]\{\dot{\varepsilon}\} \tag{10}$$

Relation (10) holds only when yielding has occurred. Thus, by introducing the following Heaviside type functions:

$$H_1(\Phi) = \begin{cases}1, & \Phi = 0\\ 0, & \Phi < 0\end{cases} \qquad H_2(\Phi) = \begin{cases}1, & (\partial\Phi/\partial\boldsymbol{\sigma}):d\boldsymbol{\sigma} \geq 0\\ 0, & (\partial\Phi/\partial\boldsymbol{\sigma}):d\boldsymbol{\sigma} < 0\end{cases} \tag{11}$$

a single relation is established for the plastic multiplier in the whole stress space, which is the main intervention of the Bouc – Wen model:

$$\dot{\lambda} = H_1 H_2 \left(-\left\{ \frac{\partial \Phi}{\partial \{\alpha\}} \right\}^{\mathrm{T}} \{G\} + \left\{ \frac{\partial \Phi}{\partial \{\sigma\}} \right\}^{\mathrm{T}} [C] \left\{ \frac{\partial \Phi}{\partial \{\sigma\}} \right\} \right)^{-1} \left\{ \frac{\partial \Phi}{\partial \{\sigma\}} \right\}^{\mathrm{T}} [C]\{\dot{\varepsilon}\} \tag{12}$$

To derive the Bouc-Wen relations, the two Heaviside functions are smoothened using the following expressions:

$$H_1 = \left| \Phi / \Phi_0 \right|^N, \quad N \geq 2 \tag{13}$$

and (since there is no plastic deformation during unloading and $\dot{\lambda} = 0$):

$$H_2 = H\left(\left\{ \frac{\partial \Phi}{\partial \{\sigma\}} \right\}^{\mathrm{T}} \{\dot{\sigma}\} \right) = \frac{1}{2} + \frac{1}{2} sign\left(\left\{ \frac{\partial \Phi}{\partial \{\sigma\}} \right\}^{\mathrm{T}} [C]\{\dot{\varepsilon}\} \right) \tag{14}$$

Finally, using eq. (3) and eq. (12), the following Bouc – Wen model is derived:

$$\{\dot{\varepsilon}^{pl}\} = \underbrace{\left| \frac{\Phi}{\Phi_0} \right|^N}_{H_1} \underbrace{\left(\frac{1}{2} + \frac{1}{2} sign\left(\left\{ \frac{\partial \Phi}{\partial \{\sigma\}} \right\}^{\mathrm{T}} [C]\{\dot{\varepsilon}\} \right) \right)}_{H_2} [R]\{\dot{\varepsilon}\} \tag{15}$$

where the interaction matrix $[R]$ is expressed as:

$$[R] = \left(-\left\{ \frac{\partial \Phi}{\partial \{\alpha\}} \right\}^{\mathrm{T}} \{G\} + \left\{ \frac{\partial \Phi}{\partial \{\sigma\}} \right\}^{\mathrm{T}} [C] \left\{ \frac{\partial \Phi}{\partial \{\sigma\}} \right\} \right)^{-1} \left\{ \frac{\partial \Phi}{\partial \{\sigma\}} \right\} \left\{ \frac{\partial \Phi}{\partial \{\sigma\}} \right\}^{\mathrm{T}} [C] \tag{16}$$

and determines the necessary interrelations between the plastic strain components to secure that the stresses remain on the yield surface accounting also for the hardening law (consistency condition). In this rate form the Bouc-Wen model can incorporate any yield criterion and hardening law, encapsulating all different aspects of loading and unloading phases [9], [10]. In the case where von Mises yield criterion is utilized the expressions are:

$$\Phi = \frac{1}{\sqrt{2}} \left[\left(\sigma_{xx} - \sigma_{yy} \right)^2 + \left(\sigma_{yy} - \sigma_{zz} \right)^2 + \left(\sigma_{xx} - \sigma_{zz} \right)^2 + 6\left(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{xz}^2 \right) \right]^{1/2}, \quad \Phi_0 = \sigma_{yield} \tag{17}$$

It is worth noting that Bouc-Wen model can be easily extended to account for stiffness degradation, strength deterioration and pinching phenomena [7] by introducing additional parameters and thus addressing a more realistic degrading behavior, which usually accompanies cyclic excitation.

3 THE MITC9 SHELL ELEMENT

Considering the shell in Figure 1, let ξ, η be two curvilinear coordinates in the middle plane of the shell and ζ a linear coordinate in the thickness direction. If further we assume that ξ, η, ζ vary between *-1* and *1* on the respective faces of the element, we can write a relationship between the Cartesian coordinates of any point of the shell and the curvilinear coordinates in the form [1]:

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \sum_{i=1}^{9} N_i(\xi,\eta)\frac{(1+\zeta)}{2}\begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix}_{top} + \sum_{i=1}^{9} N_i(\xi,\eta)\frac{(1-\zeta)}{2}\begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix}_{bottom} \tag{18}$$

where $N_i(\xi,\eta)$ are the interpolation functions. It should be noted that the coordinate direction ζ is only approximately normal to the middle surface.

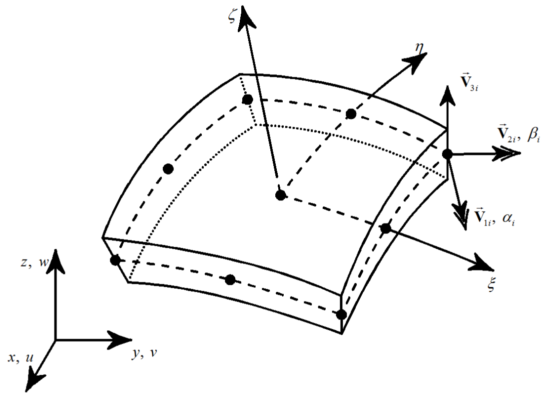


Figure 1: 9-node MITC shell element

Relation (18) can be rewritten in a form specified by the 'vector' connecting the upper and lower points (i.e. a vector of length equal to the shell thickness t_i) and the mid-surface coordinates:

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \sum_{i=1}^{9} N_i(\xi,\eta)\begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix}_{mid} + \sum_{i=1}^{9} N_i(\xi,\eta)\frac{\zeta}{2}\vec{\mathbf{V}}_{3i} \quad , \quad \vec{\mathbf{V}}_{3i} = \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix}_{top} - \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix}_{bottom} \tag{19}$$

As the strains in the direction normal to the mid-surface are assumed to be negligible, the displacement field throughout the element will be taken to be uniquely defined by the three Cartesian components of the mid-surface node displacement *i* and two rotations of the nodal vector $\vec{\mathbf{V}}_{3i}$ about orthogonal directions normal to it. If two such orthogonal directions are given by vectors $\hat{\mathbf{v}}_{1i}$ and $\hat{\mathbf{v}}_{2i}$, (of unit magnitude) with corresponding (scalar) rotations α_i and β_i, we can write, dropping the suffix 'mid' of equation (19):

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \sum_{i=1}^{9} N_i(\xi,\eta)\begin{Bmatrix} u_i \\ v_i \\ w_i \end{Bmatrix} + \sum_{i=1}^{9} N_i(\xi,\eta)\zeta\frac{t_i}{2}[-\hat{\mathbf{v}}_{2i}, \quad \hat{\mathbf{v}}_{1i}]\begin{Bmatrix} \alpha_i \\ \beta_i \end{Bmatrix} \tag{20}$$

where *u*, *v* and *w* are displacements in the directions of the global *x*, *y* and *z* axes and t_i the thickness of the shell at nodal point *i*.

From the infinite vector directions normal to a given direction that can be generated, a particular scheme is followed to ensure a unique definition. If $\hat{\mathbf{j}}$, for instance, is the unit vector along the *y* axis, $\vec{\mathbf{V}}_1 = \hat{\mathbf{j}} \times \vec{\mathbf{V}}_3$ makes the vector $\vec{\mathbf{V}}_1$ perpendicular to the plane defined by the direction $\vec{\mathbf{V}}_3$ and the *y* axis. As $\vec{\mathbf{V}}_2$ has to be orthogonal to both $\vec{\mathbf{V}}_3$ and $\vec{\mathbf{V}}_1$, $\vec{\mathbf{V}}_2 = \vec{\mathbf{V}}_3 \times \vec{\mathbf{V}}_1$. To obtain unit vectors in the three directions, $\vec{\mathbf{V}}_1$, $\vec{\mathbf{V}}_2$ and $\vec{\mathbf{V}}_3$ are simply divided by their scalar lengths, giving the unit vectors $\hat{\mathbf{v}}_{1i}$, $\hat{\mathbf{v}}_{2i}$ and $\hat{\mathbf{v}}_{3i}$. On occasions the direction of the *y* axis and $\vec{\mathbf{V}}_3$ may coincide, $\vec{\mathbf{V}}_1 = \hat{\mathbf{i}}$.

Coming to the important feature of the MITC9 shell element formulation [4], a mixed interpolation scheme is used so as to render the resulting element relatively distortion – insensitive. In the natural coordinate system of the shell element, the covariant base vectors are defined as:

$$\{g_\xi\} = \frac{\partial\{x\}}{\partial\xi}, \quad \{g_\eta\} = \frac{\partial\{x\}}{\partial\eta}, \quad \{g_z\} = \frac{\partial\{x\}}{\partial\zeta} \tag{21}$$

where $\{x\}$ is the vector of coordinates $\{x \quad y \quad z\}^T$ (eq. (19)). In the natural system, the strain tensor is expressed in terms of covariant tensor components and contravariant base vectors as:

$$\boldsymbol{\varepsilon} = \tilde{\varepsilon}_{ij}\mathbf{g}^i\mathbf{g}^j, \qquad i,j = \xi,\eta,\zeta \tag{22}$$

where the tilde (~) indicates that the tensor components are in the convected coordinate system. The evaluation of these components is achieved by using the linear terms of the relation for the strain components in terms of the base vectors:

$$\tilde{\varepsilon}_{ij} = \frac{1}{2}\left[\left\{\left\{\frac{\partial\{x\}}{\partial\xi_i}\right\}+\left\{\frac{\partial\{u\}}{\partial\xi_i}\right\}\right\}^T\left\{\left\{\frac{\partial\{x\}}{\partial\xi_j}\right\}+\left\{\frac{\partial\{u\}}{\partial\xi_j}\right\}\right\}-\left\{\frac{\partial\{x\}}{\partial\xi_i}\right\}^T\left\{\frac{\partial\{x\}}{\partial\xi_j}\right\}\right]$$

$$\tilde{\varepsilon}_{ij}\Big|_{linear} = \left[\left\{\frac{\partial\{u\}}{\partial\xi_i}\right\}^T\left\{\frac{\partial\{x\}}{\partial\xi_j}\right\}+\left\{\frac{\partial\{u\}}{\partial\xi_j}\right\}^T\left\{\frac{\partial\{x\}}{\partial\xi_i}\right\}\right] = \frac{1}{2}\left[\left\{\frac{\partial\{u\}}{\partial\xi_i}\right\}^T\{g_j\}+\left\{\frac{\partial\{u\}}{\partial\xi_j}\right\}^T\{g_i\}\right]$$

$$\xi_i = \xi,\eta,\zeta \quad , \quad i,j = \xi,\eta,\zeta \tag{23}$$

In the mixed interpolation, the in-layer ($\tilde{\varepsilon}_{\xi\xi}$, $\tilde{\varepsilon}_{\eta\eta}$, $\tilde{\varepsilon}_{\xi\eta}$) and transverse shear ($\tilde{\varepsilon}_{\xi\zeta}$, $\tilde{\varepsilon}_{\eta\zeta}$) strain components are interpolated independently and these interpolations are tied to the usual displacements interpolations. The interpolation used for the evaluation of the covariant strain component $\tilde{\varepsilon}_{ij}(\xi,\eta,\zeta)$ is [3]:

$$\tilde{\varepsilon}_{ij}(\xi,\eta,\zeta) = \sum_{k=1}^{n} h_k(\xi,\eta)\,\tilde{\varepsilon}_{ij}^k(\xi,\eta,\zeta) \tag{24}$$

where $h_k(\xi,\eta)$ are interpolation functions and $\tilde{\varepsilon}_{ij}^{k}(\xi,\eta,\zeta)$ is the covariant strain component evaluated be eq. (23) at tying point *k* (Figure 2).

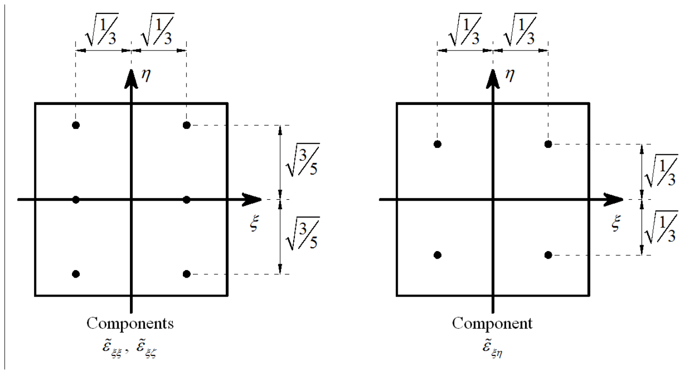


Figure 2: Tying points used for the strain components of the MITC9 shell element. Tying points for the components $\tilde{\varepsilon}_{\eta\eta}$ and $\tilde{\varepsilon}_{\eta\zeta}$ are implied by symmetry

Finally, combining equations (22), (24) and (23), the following relation is derived:

$$\underbrace{\{\varepsilon\}}_{(6\times1)} = \underbrace{[B]}_{(6\times45)}\underbrace{\{d\}}_{(45\times1)} \tag{25}$$

The stress – strain law must contain the shell assumption that the stress normal to the shell surface is zero. Thus:

$$\{\tau\} = [C]\{\varepsilon\}$$

$$\{\tau\} = \{\tau_{xx} \quad \tau_{yy} \quad \tau_{zz} \quad \tau_{xy} \quad \tau_{yz} \quad \tau_{zx}\}^T, \qquad \{\varepsilon\} = \{\varepsilon_{xx} \quad \varepsilon_{yy} \quad \varepsilon_{zz} \quad \gamma_{xy} \quad \gamma_{yz} \quad \gamma_{zx}\}^T$$

$$[C] = [Q]^T \left(\frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 & 0 & 0 & 0 \\ \nu & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_s\frac{1-\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_s\frac{1-\nu}{2} \end{bmatrix} \right) [Q] \tag{26}$$

where *k* is the shear correction factor of $5/6$ and $[Q]$ represents a matrix that transforms the stress – strain law from a ξ, η, ζ Cartesian shell-alligned system to the global Cartesian coordinate system.

4 STIFFNESS AND HYSTERETIC MATRICES

The elastic deformation field is extended by introducing an additional vector of hysteretic degrees of freedom which herein are the plastic strains at Gauss points of all layer interfaces [8]:

$$\{z\}=\left\{\underbrace{\left\{\overbrace{\left\{\varepsilon^{pl}\right\}_{1,1}^{T}}^{\text{interface }1},\ \overbrace{\left\{\varepsilon^{pl}\right\}_{1,2}^{T}}^{\text{interface }2},\ \cdots\ \overbrace{\left\{\varepsilon^{pl}\right\}_{1,n}^{T}}^{\text{interface }n}\right\}}_{Gauss\text{-}Point\ \ 1},\ \cdots,\ \underbrace{\left\{\left\{\varepsilon^{pl}\right\}_{9,1}^{T},\ \left\{\varepsilon^{pl}\right\}_{9,2}^{T},\ \cdots\ \left\{\varepsilon^{pl}\right\}_{9,n}^{T}\right\}}_{Gauss\text{-}Point\ \ 9}\right\}^{T} \tag{27}$$

where $\left\{\varepsilon^{pl}\right\}_{i,j}$ is the plastic component of the strain vector (eq. (27)) at the i^{th} Gauss point of the j^{th} interface. At this point, a hysteretic linear interpolation field $\left[N_{pl}\right]$ can be considered utilizing appropriate shape functions so that:

$$\underset{6\times1}{\left\{\varepsilon^{pl}\right\}}=\underset{6\times(54n)}{\left[N_{pl}\right]}\ \underset{(54n)\times1}{\{z\}} \tag{28}$$

where $n-1$ is the total number of layers.

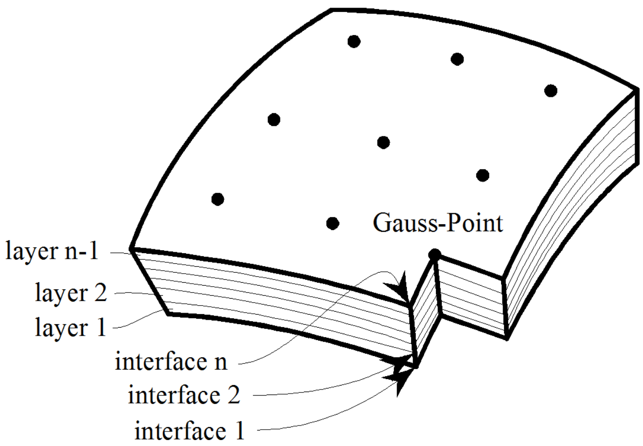


Figure 3: A layered MITC9 shell

The expression for the principle of virtual work in this case is written as:

$$\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\{\bar{\varepsilon}\}^{\mathrm{T}}\{\sigma\}\det[J]\,d\xi\,d\eta\,dz=\left\{\bar{d}\right\}^{T}\{R_C\} \tag{29}$$

where $\left\{\bar{d}\right\}$ are the virtual displacements and $\{\bar{\varepsilon}\}$ are the corresponding virtual strains and by means of relation (8) the principle of virtual work (29) is expressed as:

$$\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\{\bar{\varepsilon}\}^{\mathrm{T}}[C]\left(\{\varepsilon\}-\left\{\varepsilon^{pl}\right\}\right)\det[J]\,d\xi\,d\eta\,dz=\left\{\bar{d}\right\}^{T}\{R_C\} \tag{30}$$

Substituting relations (27) and (28) into relation (30) the following expression is obtained:

$$\{\bar{d}\}^T \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}[B]^{\mathrm{T}}[C]\left([B]\{D\}-\left[N_{pl}\right]\{z\}\right)\det[J]\,d\xi\,d\eta\,dz=\{\bar{d}\}^T\{R_C\}\Leftrightarrow$$
$$\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}[B]^{\mathrm{T}}[C][B]\{d\}\det[J]\,d\xi d\eta\,dz-\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}[B]^{\mathrm{T}}[C]\left[N_{pl}\right]\{z\}\det[J]\,d\xi d\eta\,dz=\{R_C\} \tag{31}$$

and finally the following constitutive equation is obtained at the element level:

$$[k_e]\{d\}-[k_h]\{z\}=\left[\begin{array}{c:c}\underset{(45\times45)}{[k_e]} & \underset{(45\times(54n))}{-[k_h]}\end{array}\right]\left\{\begin{array}{c}\{d\}\\ \hline \{z\}\end{array}\right\}=\{R_c\} \tag{32}$$

where $n-1$ is the total number of layers, $[k_e]$ is the MITC9 elastic and $[k_h]$ the herein introduced hysteretic stiffness matrix. The number of columns of the hysteretic matrix $[k_h]$ corresponds to six (6) components of strain at each of the nine (9) Gauss points for the n interfaces of the thickness of the shell.

The additional unknown vector $\{z\}$, containing all plastic strains at all Gauss points of all interfaces, follows an evolutionary equation of Bouc-Wen type given in relation (15) independently for every six component plastic strain vector at every particular Gauss point.

From the aforementioned it becomes evident that the proposed formulation can be used also for other types of elements to incorporate directly the hysteretic behavior.

5 STATE EQUATIONS - SOLUTION PROCEDURE

The elemental stiffness and hysteretic matrices derived using eq. (32) are assembled to form the structural stiffness matrix $[K_S]$ and the structural hysteretic matrix $[H_S]$. Moreover, n_f is the number of total degrees of freedom of the structure, $(n_{\text{int}}-1)$ the number of layers into which the shell is subdivided and n_{hys} is the number of hysteretic degrees of freedom. It holds that $n_{hys}=n_{elem}\times(54\times n_{\text{int}})$, where n_{elem} is the number of the shell elements of the structure. The equation of motion is then expressed as:

$$[M_s]\{\ddot{U}\}+[C_s]\{\dot{U}\}+[K_s]\{U\}-[H_s]\{Z\}=\{P(t)\} \tag{33}$$

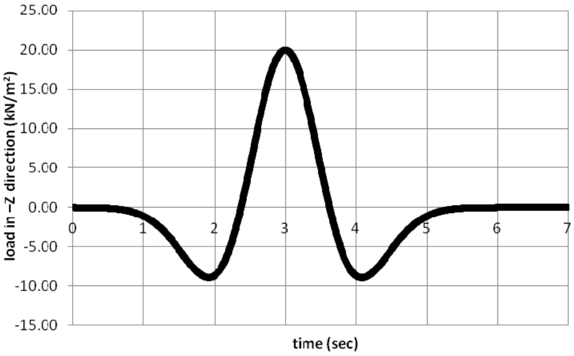
where $[M_s]$, $[C_s]$, $[K_s]$ are the mass, viscous damping and stiffness square symmetric $(n_f\times n_f)$ matrices of the structure respectively [5], while $[H_s]$ is in general the orthogonal global hysteretic $(n_f\times n_{hys})$ matrix of the structure and $\{Z\}$ is the $(n_{hys}\times1)$ vector of hysteretic degrees of freedom which contains the $(54n_{\text{int}}\times1)$ vectors $\{z\}$ of all elements of the structure and $\{P(t)\}$ is the $(n_f\times1)$ vector of external forces. All the above matrices are evaluated only at the beginning of the analysis requiring no updating during the subsequent analysis procedure and the lineal equations of motion can be solved following Newmark's method.

In addition to the linear equations of motion (eq. (33)), the uncoupled non-linear evolution equations of the hysteretic degrees of freedom (eq. (15)) for each Gauss point at every layer for all the elements of structure are required. The system of first order nonlinear differential evolution equations can be solved using Runge – Kutta integrators or predictor-corrector methods, such as the Livermore family of solvers, allowing for robust and unconditionally stable solutions.

Plastic flow rules are incremental in nature and standard FEM solution procedures have to follow small equilibrium steps to trace their path. Accuracy is needed at both the flow rule within an increment and keeping the solution on the yield surface. The solution is finally equilibrated only at the end of each increment after a number of equilibrium iterations. Therefore, classical elastoplastic solution procedures are based on incremental predictor/corrector schemes, which eventually accumulate some error. This is attributed to errors in the integration of the flow rule and their relation to the complete incremental / iterative solution procedure. A basic advantage of the proposed method is that the load is handled through the system of first order differential equations. The proposed formulation provides the nodal displacements, the elastic and plastic strains and the stresses at all Gauss points of every layer that satisfy the inelastic constitutive relations and equilibrium without any additional iterative process. Therefore, by solving in steps the system of differential equations that includes the evolution equations, the scheme stays always on the yield function and satisfies the flow rule by default and therefore the local iterations of radial return or backward Euler method are avoided in solving for the equilibrium of the system.

6 NUMERICAL EXAMPLE

A Fortran code was developed to implement and subsequently test the efficiency of the proposed formulation. In this example the Scordelis – Lo roof is subjected to a time varying load (Ricker pulse) with maximum intensity $q=20kN/m^2$, uniformly distributed on the surface of the shell in –Z direction. The thickness of the roof is $0.25m$, with the indicated boundary conditions. The roof is discretized into $9x13$ 9-node MITC9 shell elements and the computational model consists of 10 layers and 513 nodes. The material parameters are $E=432GPa$, $E_t=43.2GPa$, $v=0.0$ and $\sigma_y=24MPa$, where E is the Young's modulus, E_t is the tangent modulus, v is the Poisson's ratio and σ_y is the yield stress. The density of the material is $7.85\ Mgr/m^3$ and linear kinematic hardening is considered. In addition, the material follows the von – Mises yield criterion.



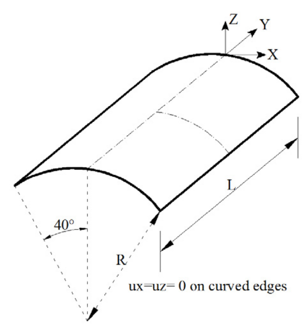


Figure 4: Geometry ($L=50m$, $R=25m$), boundary conditions and loading of Scordelis-Lo roof

In Figure 5 the response of the roof is plotted in terms of vertical displacement at the node A versus time. In the same figure the derived displacement history curve is compared against the one obtained using Abaqus. In Figure 6 and Figure 7 the vertical component of velocity and acceleration history of node A is plotted respectively.

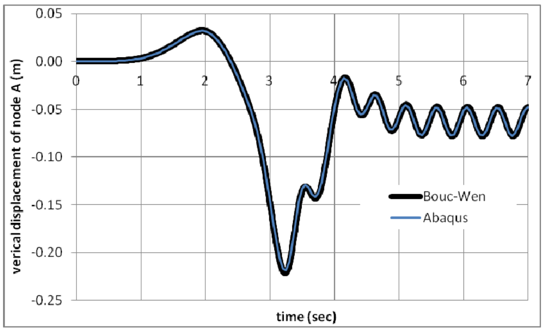


Figure 5: Vertical displacement of node A (m)

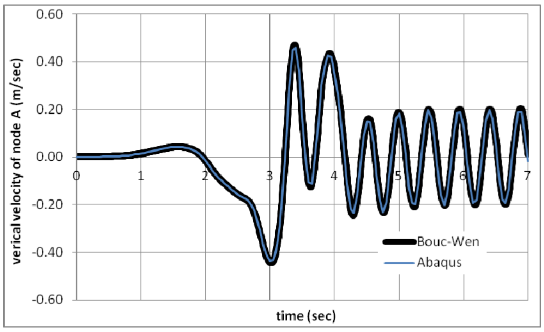


Figure 6: Vertical velocity of node A (m/sec)

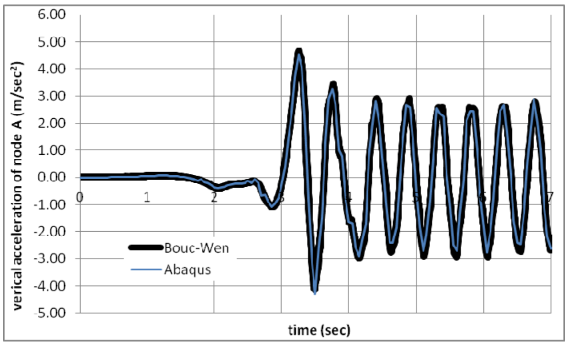


Figure 7: Vertical acceleration of node A (m/sec^2)

From all the previous figures and comparisons, it is apparent that the solution obtained based on the proposed formulation agrees well with the solution obtained using Abaqus and turns out as computationally advantageous for the same accuracy.

7 CONCLUSIONS

A hysteretic MITC9 shell finite element is developed suitable for the inelastic static and dynamic analysis of shell structures. A smooth rate independent hysteretic model of Bouc-Wen type is incorporated in the constitutive relations of the standard Finite Element formulation yielding not only the elastic but also and hysteretic element matrices. Plastic strains at all shell layer interfaces are introduced as additional unknowns together with corresponding evolution equations. The system of governing equations is solved numerically following a state space formulation.

The basic advantage of the proposed method relies on the fact that the response is handled through the system of first order differential equations. This provides the nodal displacements, the elastic and plastic strains and the stresses at all Gauss points of every layer that satisfy the inelastic constitutive relations and equilibrium without any additional iterative process. Therefore, by solving the system of differential equations numerically, the scheme stays always on the yield function and satisfies the flow rule by definition. Consequently, the local iterations of Newton – Raphson method are avoided, at the expense of the numerical solution of first order evolution equations for the introduced additional hysteretic unknowns. The proposed formulation utilizes the inelastic constitutive relation in the principle of virtual work in a separable form distinguishing the elastic and hysteretic part. This results into structural matrices that are evaluated only once, at the beginning of the analysis procedure. The proposed formulation directly accounts for inelasticity in a natural way by solving in coupled form the linear equilibrium equations together with the non-linear evolution equations. This avoids the inconsiderate elastic predictions, which are belatedly followed by plastic corrections. For these reasons, the proposed formulation turns out computationally more efficient for the same accuracy as compared to standard methods especially in dynamical problems.

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