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ON A CONSISTENT APPLICATION OF NEWTON'S LAW TO MECHANICAL SYSTEMS WITH MOTION CONSTRAINTS

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Abstract. A new theoretical approach is presented for deriving a set of equations of motion for a class of mechanical systems subjected to holonomic and/or nonholonomic constraints. The main idea is to consider the equations describing the action of the constraints as an integral part of the overall process leading to the equations of motion of the system. As a consequence of this approach, a set of coupled second order ordinary differential equations is derived for both the generalized coordinates and the dynamic Lagrange multipliers associated to the motion constraints automatically. This leads to considerable theoretical advantages, avoiding problems related to systems of differential-algebraic equations or penalty formulations. Apart from its theoretical value and elegance, this approach is expected to bring significant benefits in the field of numerical integration in multibody dynamics and control.

1 INTRODUCTION

In engineering applications, the configuration of a system is described usually by a set of generalized coordinates. Frequently, a minimum number of them is chosen. However, in many cases it is beneficial to use more coordinates than those actually needed. Then, extra equations are also introduced, representing the effect of motion constraints [1, 2]. These equations complement the set of second order ordinary differential equations (ODEs), arising by application of the law of motion to all components of the system. This gives rise to a set of differential-algebraic equations (DAEs), leading to severe difficulties during their numerical solution [3]. A comprehensive review on these problems can be found in [4].

The main objective of this work is to present a new set of equations of motion for mechanical systems subjected to scleronomic equality constraints. The new approach is based on some fundamental concepts of differential geometry and treats both holonomic and nonholonomic constraints by considering them as part of the overall process of deriving the equations of motion. This leads to an assignment of inertia, damping and stiffness properties to the action of the constraints. As a result, the equations of motion are second order ODEs in both the generalized coordinates and the Lagrange multipliers. Consequently, there is no need to introduce artificial parameters for scaling and stabilization, as required by DAE or penalty based formulations. In addition, the geometric properties of the original manifold are kept unchanged by the additional constraints. This preserves the geometric properties of special curves of the manifold employed and leads to major advantages compared to previous work in the field of computational Multibody Dynamics [1, 4].

The organization of this paper is as follows. First, some useful concepts of differential geometry, related to dynamics of constrained mechanical systems, are briefly summarized in the following section. Then, two conditions on the metric and connection of two manifolds, describing the motion of a system with a different number of constraints, are used so that the form of Newton's law of motion remains invariant [5]. This sets up the ground for a consistent derivation of the equations of motion, which is a task completed in Sects. 4 and 5. Finally, the most important conclusions are summarized in the last section.

2 MOTION OF A DYNAMICAL SYSTEM ON A MANIFOLD

The configuration of a mechanical system can frequently be determined by a finite number of generalized coordinates, $q^1,\ldots,q^n$. Then, the motion of the system is represented by the motion of a point along a curve $\gamma=\gamma(t)$ on an n-dimensional manifold M, known as the configuration space [6]. Moreover, the tangent vector $\underline{v}=d\gamma/dt$ to curve γ at a point p belongs to an n-dimensional vector space T_pM, the tangent space of the manifold. Therefore, if $\mathfrak{B}=\{\underline{e}_1 \quad \ldots \quad \underline{e}_n\}$ is a basis, any element $\underline{u}$ of T_pM can be expressed in the form

$$\underline{u}=\sum_{i=1}^{n} u^i\,\underline{e}_i = u^i\,\underline{e}_i\,,$$

by adopting the usual summation convention for repeated indices [7]. In addition, the rate of change of a vector field $\underline{u}(t)$ on M along a curve with tangent vector $\underline{v}$ is determined by the covariant differential of $\underline{u}(t)$ along $\underline{v}$, having the form

$$\nabla_{\underline{v}}\underline{u}(t)=(\dot{u}^k+\Lambda^k_{ij}v^iu^j)\underline{e}_k\,, \tag{1}$$

where ∇ represents a mathematical entity known as affine connection of the manifold, while

$$\dot{u}^i=\tfrac{d}{dt}u^i(t) \quad \text{and} \quad v^i=\dot{q}^i=\tfrac{d}{dt}q^i(t)\,.$$

The components Λ^i_{jk} of connection ∇ in basis $\mathfrak{B}$ of T_pM, known as affinities, are defined by

$$\nabla_{\underline{e}_j}\underline{e}_k=\Lambda^i_{jk}\underline{e}_i\,. \tag{2}$$

At every point p of a manifold, one can also define the dual or cotangent space T_p^*M. Then, at any point p and for any vector $\underline{u}$, a covector $\underset{\sim}{u}^*$ may be identified and vice versa. In dynamics, this correspondence is commonly established through the dual product

$$\underset{\sim}{u}^*(\underline{w})\equiv\langle\underline{u},\underline{w}\rangle,\quad \forall\underline{w}\in T_pM\,, \tag{3}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product of T_pM [6]. In this way, to each basis $\{\underline{e}_i\}$ (with $i=1,\ldots,n$) of T_pM, a dual basis $\{\underset{\sim}{e}^i\}$ can be established for T_p^*M by employing the condition

$$\underset{\sim}{e}^i(\underline{e}_j)=\delta^i_j\,, \tag{4}$$

where δ^i_j is a Kronecker's delta [7]. Then, the covariant differential of a covector field $\underset{\sim}{u}^*(t)$ on M along a vector $\underline{v}$ of T_pM is evaluated in the form

$$\nabla_{\underline{v}}\underset{\sim}{u}^*(t)=(\dot{u}_k-\Lambda^j_{ik}v^iu_j)\underset{\sim}{e}^k \tag{5}$$

Finally, determination of the true path of motion on a manifold is based on application of Newton's second law. Starting with a single particle, this law appears in the form

$$\nabla_{\underline{v}} \underset{\sim}{p}^* = \underset{\sim}{f}^* . \tag{6}$$

The covector $\underset{\sim}{p}^*$ stands for the generalized momentum of the particle, while $\underset{\sim}{f}^*$ represents the resultant of the external forcing on the particle [5, 6]. Generalizing Eq. (6), the motion on a configuration manifold M is governed by Newton's law in the form

$$\nabla_{\underline{v}} \underset{\sim}{p}_M^* = \underset{\sim}{f}_M^* , \tag{7}$$

where the covector $\underset{\sim}{f}_M^* = f_i \underset{\sim}{e}^i$ represents the resultant external force. Moreover, the generalized momentum is defined as the covector corresponding to the velocity vector, i.e., $\underset{\sim}{p}^* \equiv \underset{\sim}{v}^*$. Then, if $\underline{v} = v^i \underline{e}_i$ and $\underset{\sim}{p}^* = p_i \underset{\sim}{e}^i$, application of the definition (3) leads to

$$p_i = g_{ij} v^j , \tag{8}$$

where the quantities

$$g_{ij} = \langle \underline{e}_i , \underline{e}_j \rangle \tag{9}$$

are components of the metric tensor at point p. These quantities can be selected to coincide with the elements of the mass matrix of the system, defined through the kinetic energy. Frequently, it is useful to think of Newton's law (7) as a map, defined by the covectors

$$\underset{\sim}{h}_M^* \equiv \nabla_{\underline{v}} \underset{\sim}{p}_M^* - \underset{\sim}{f}_M^* . \tag{10}$$

Among them, the zero covector satisfies Newton's law and will be called a Newton covector.

3 APPLICATION OF NEWTON'S LAW TO CONSTRAINED SYSTEMS

Next, assume that the motion of the system examined is subjected to an additional set of k holonomic and/or nonholonomic scleronomic constraints, cast in the form

$$A(\underline{q})\underline{v} = \underline{0} , \tag{11}$$

where $A = [a_i^R]$ is a $k \times n$ matrix and $\underline{v} = \dot{\underline{q}}$. When all these constraints are holonomic, the equation of constraints can be put in the algebraic form

$$\underline{\phi}(\underline{q}) = \underline{0} . \tag{12}$$

After imposing this additional set of constraints, the motion of the system takes place on a curve $\gamma_A(t)$ of another manifold M_A, with dimension $m = n - k$. Let $\{\theta^\alpha\}$ ($\alpha = 1, \ldots, m$) be a set of generalized coordinates in a neighborhood of a point p_A, related to point p of M through a mapping defined by the constraints, which will become more explicit next. Furthermore, let $\{\underline{e}_\alpha\}$ be a basis of the tangent space $T_{p_A} M_A$ and $\{\underset{\sim}{e}^\alpha\}$ be the corresponding basis of the dual space $T_{p_A}^* M_A$. If the constraints are linearly independent, the generalized velocities are split as

$$\underline{v} = (\underline{v}_D^T \quad \underline{v}_I^T)^T ,$$

where $\underline{v}_D$ and $\underline{v}_I \equiv \dot{\underline{\theta}}$ include the dependent and independent generalized velocities, respectively. Then, by first partitioning the constraint matrix A, it eventually turns out that

$$\underline{v} = N \dot{\underline{\theta}} , \tag{13}$$

where N is an $n \times m$ matrix depending on A. This defines a linear transformation E_S, by

$$E_S = N^i_{\cdot\alpha} \underset{\sim}{e}_i \otimes \underset{\sim}{e}^\alpha \,, \tag{14}$$

where symbol $\otimes$ represents the classical tensor product [7]. It also defines the dual operator

$$E_{SD} = N^{\cdot i}_{\alpha} \underset{\sim}{e}^\alpha \otimes \underset{\sim}{e}_i \,. \tag{15}$$

The lower case Latin indices vary from 1 to n. Lower case Greek indices vary from 1 to m.

The linear mapping E_{SD} presents a suitable vehicle for transferring the law of motion from M to M_A. In particular, it is first required that the law governing the motion on the manifold M_A preserves the form of Newton's law expressed by Eq. (7). This means that

$$\nabla_{\underset{\sim}{v}_A} \underset{\sim}{p}^*_A = \underset{\sim}{f}^*_A \,. \tag{16}$$

In analogy to Eq. (10), one can then define a class of Newton covectors on $T^*_{p_A} M_A$ with form

$$\underset{\sim}{h}^*_A \equiv \nabla_{\underset{\sim}{v}_A} \underset{\sim}{p}^*_A - \underset{\sim}{f}^*_A \,. \tag{17}$$

Then, it is required that the following condition is also satisfied

$$\underset{\sim}{h}^*_A = E_{SD} \underset{\sim}{h}^*_M \,. \tag{18}$$

Based on these, it was shown in a previous study [5] that if the motion on manifold M is governed by Newton's law in the form (7), this form remains the same even on the new manifold M_A, provided that the metric and the affinities on M_A satisfy the conditions

$$g_{\alpha\beta} = N^i_\alpha g_{ij} N^j_\beta \tag{19}$$

and

$$(\Lambda^\rho_{\gamma\alpha} g_{\rho\beta} - N^i_{\alpha,\gamma} N^j_\beta g_{ij} - N^i_\alpha N^j_\beta N^k_\gamma \Lambda^m_{ji} g_{mk}) v^\beta v^\gamma = 0 \,, \tag{20}$$

where $\Lambda^\gamma_{\alpha\beta}$ and $g_{\alpha\beta}$ are the components of the connection and metric tensor at a point of M_A.

4 OPERATORS ACTING BETWEEN TANGENT AND COTANGENT SPACES

4.1 Definition of operators acting between tangent spaces

A key step in the process developed is that each of the k equations of constraints is considered separately. Specifically, the constraints (11) are viewed as dual products

$$\dot{\psi}^R \equiv \underset{\sim}{a}^R(\underset{\sim}{v}) = 0 \,, \qquad (R = 1, \ldots, k), \tag{21}$$

where the quantity $\underset{\sim}{a}^R$ represents the R-th row of matrix A with elements $a^R_{\cdot i}$ and is treated as a covector of $T^*_p M$. The last relation provides the foundation to define the linear operator

$$T_R = a^R_{\cdot i} \underset{\sim}{e}_R \otimes \underset{\sim}{e}^i \,, \tag{22}$$

representing a mapping from $T_p M$ to a single-dimensional space $W_R \equiv T_{p_R} M_R$, with basis vector $\underset{\sim}{e}_R$. The underlying manifold M_R and its point p_R are also defined by the action of the R-th constraint in a way that will be clarified in Sect. 5. From hereon, an upper case Latin index varies from 1 to k and does not obey the summation convention on repeated indices.

By definition, operator T_R takes vectors of $T_p M$ and returns vectors of W_R. In particular,

$$T_R \underset{\sim}{e}_i = a^R_i \underset{\sim}{e}_R \,. \tag{23}$$

In general, the operator T_R is surjective. Therefore, it possesses a null space, say H_R, which is a subspace of $T_p M$ [7]. Moreover, it accepts a right inverse, say S_R, which is an injective linear mapping from W_R to a single-dimensional subspace V_R of T_pM, so that

$$T_R S_R = I_R , \tag{24}$$

where I_R is the identity operator in W_R. In addition,

$$S_R = c^i_{\cdot R} \underline{e}_i \otimes \underset{\sim}{e}^R \tag{25}$$

and consequently

$$S_R \underline{e}_R = c^i_R \underline{e}_i \equiv \underline{c}_R , \tag{26}$$

where vector $\underline{c}_R$ can be selected as basis of V_R. Next, application of (23) and (26) yields

$$T_R S_R \underline{\alpha}_R = \underline{\alpha}_R ,$$

which in turn leads to

$$c^i_R a^R_i = 1 . \tag{27}$$

Satisfaction of the last condition requires that the value of the components of vector $\underline{c}_R$ are

$$c^i_R = \hat{c}_R a^R_j \delta^{ji} \quad \text{with} \quad \hat{c}_R = 1/(a^R_i a^R_i) . \tag{28}$$

Finally, it is an easy task to show that the operator

$$\Pi_R \equiv S_R T_R \tag{29}$$

is a projection. In particular, it is easy to verify that $\Pi_R \underline{v} = \underline{v}_R$, with $\underline{v}_R = \alpha^R \underline{c}_R \in V_R$, showing that space V_R is the range of Π_R. In addition, it can be concluded from Eq. (29) that the null space of Π_R coincides with that of T_R. Therefore, the space T_pM can be split in the form

$$T_pM = H_R \oplus V_R , \tag{30}$$

where H_R and V_R are ($n-1$)- and one-dimensional vector subspaces of T_pM, known as the horizontal and vertical space of T_pM, respectively. Based on the projection theorem [7], this implies that any element $\underline{v}$ of T_pM can be decomposed uniquely in the form

$$\underline{v} = \underline{u}_R + \underline{v}_R , \tag{31}$$

where $\underline{u}_R \in H_R$ and $\underline{v}_R \in V_R$. Then, by definition

$$T_R \underline{u}_R = \underline{0} . \tag{32}$$

The analysis performed so far can also be used to define a global decomposition of T_pM in the presence of multiple constraints ($k \geq 2$). To achieve this, by successively applying all the constraints one can eventually arrive at the following overall splitting

$$T_pM = H_T \oplus V_S , \tag{33}$$

where H_T is an ($n-k$)- and V_S is a k-dimensional vector subspace of T_pM, defined by

$$H_T \equiv \bigcap_{R=1}^{k} H_R \quad \text{and} \quad V_S \equiv V_1 \oplus \cdots \oplus V_k , \tag{34}$$

respectively. In addition, one can create the k-dimensional Cartesian product space

$$W \equiv W_1 \times \cdots \times W_k \cong \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{k\ times} ,$$

which can be viewed as the tangent space of a k-dimensional underlying manifold

$$M_C \equiv M_1 \times \cdots \times M_k . \tag{35}$$

Using the decomposition (33), an arbitrary vector of T_pM can be expressed in the form

$$\underline{v} = \underline{u}_T + \underline{v}_S, \quad \text{with} \quad \underline{v}_S = \underline{v}_1 + \cdots + \underline{v}_k, \tag{36}$$

where $\underline{u}_T \in H_T$ and $\underline{v}_R \in V_R$ for $R = 1,\ldots,k$ (see Fig. 1). Then, the above can be used to define a composite transformation $T : T_pM \to W$ by

$$T\underline{v} = T\underline{v}_S = T(\underline{v}_1 + \cdots + \underline{v}_k) \equiv \underline{\alpha} .$$

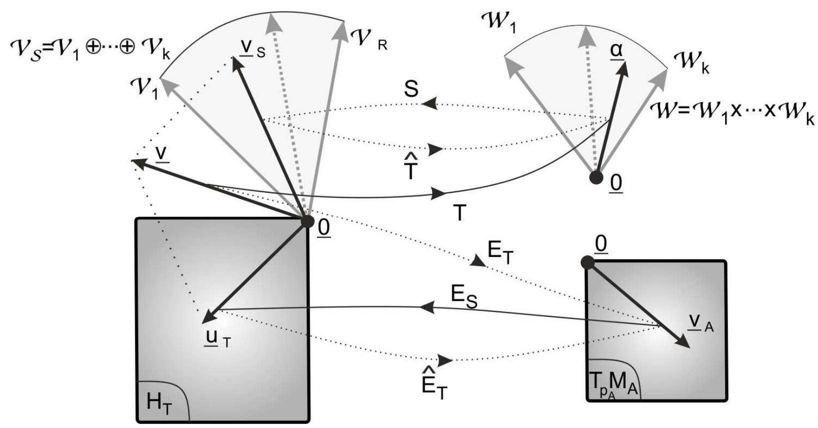

Figure 1: Transformations between the tangent spaces of manifolds M , M_A and M_C

Likewise, another composite transformation $S : W \to V_S$ can also be defined by

$$S\underline{e}_R = S_R\underline{e}_R , \tag{37}$$

so that, if $\underline{\alpha}_R$ is an element of $W_R \subset W$, then

$$S\underline{\alpha}_R = S(\alpha^R \underline{e}_R) = \alpha^R S_R \underline{e}_R = \alpha^R \underline{c}_R .$$

This means that the image of $\underline{\alpha}_R$ through S lies in $V_R \subset V_S$ only and not in all of V_S .

Next, it is easy to verify that the operator T is characterized by the following properties

$$\eta(T) = H_T , \quad \mathrm{Im}(T) = W \quad \text{and} \quad \rho(T) = k ,$$

representing null space, range and rank, respectively. The corresponding properties of S are

$$\eta(S) = \{\underline{0}\} , \quad \mathrm{Im}(S) = V_S \quad \text{and} \quad \rho(S) = k .$$

In addition, a composite bijection $\hat{T}$ from V_S to W results by elimination of the null space H_T of T through a proper projection.

To complete the picture for the tangent spaces, an operator E_S must also be defined as in Sect. 3, through Eq. (14). By construction, this operator is characterized by the properties

$$\eta(E_S) = \{\underline{0}\} , \quad \mathrm{Im}(E_S) = H_T \quad \text{and} \quad \rho(E_S) = m . \tag{38}$$

Therefore, E_S is injective, which implies that it has a left inverse [7], say $\hat{E}_T$, defined by

$$\hat{E}_T E_S = I_m \tag{39}$$

with $\eta(\hat{E}_T) = \{\underline{0}\}$, $\mathrm{Im}(\hat{E}_T) = T_{p_A}M_A$ and $\rho(\hat{E}_T) = m$. Finally, a related operator E_T can also be definaed, so that $\eta(E_T) = V_S$, $\mathrm{Im}(E_T) = T_{p_A}M_A$ and $\rho(E_T) = m$. Then, $\hat{E}_T$ can be seen as a bijection from H_T to $T_{p_A}M_A$, resulting by elimination of the null space of E_T through the projection from T_pM to H_T , defined by

$$\Pi_E = E_S E_T . \tag{40}$$

4.2 Definition of operators acting between dual spaces

An analogous picture is obtained for the cotangent spaces. In particular, two new operators, related to T_R and S_R, are defined first according to

$$T_{RD} = a_i^{\cdot R} \underset{\sim}{e}^i \otimes \underline{e}_R \quad \Rightarrow \quad T_{RD} \underset{\sim}{e}^R = a_i^{\cdot R} \underset{\sim}{e}^i \equiv \underset{\sim}{c}^R \tag{41}$$

so that the covector $\underset{\sim}{c}^R$ provides a basis to the single-dimensional space V_R^* and

$$S_{RD} = c_R^{\cdot i} \underset{\sim}{e}^R \otimes \underline{e}_i \quad \Rightarrow \quad S_{RD} \underset{\sim}{e}^i = c_R^{\cdot i} \underset{\sim}{e}^R . \tag{42}$$

In general, S_{RD} is surjective. Combination of (41) and (42) leads to

$$S_{RD} T_{RD} = I_R ,$$

showing that T_{RD} is the right inverse of S_{RD}. Furthermore, it can be verified that the operator

$$\Pi_{RD} \equiv T_{RD} S_{RD}$$

is a projection from $T_p^* M$ to V_R^*. Therefore, the dual space $T_p^* M$ can be split in the form

$$T_p^* M = H_R^* \oplus V_R^* , \tag{43}$$

for each $R = 1,\ldots,k$, where H_R^* is an ($n-1$)-dimensional subspace of $T_p^* M$, representing the null space of Π_{RD}. Based on (43), any element $\underset{\sim}{v}^*$ of $T_p^* M$ can be decomposed uniquely as

$$\underset{\sim}{v}^* = \underset{\sim}{u}_R^* + \underset{\sim}{v}_R^* , \tag{44}$$

where $\underset{\sim}{u}_R^* \in H_R^*$ and $\underset{\sim}{v}_R^* \in V_R^*$, with

$$S_{RD} \underset{\sim}{u}_R^* = \underset{\sim}{0} . \tag{45}$$

Successive application of S_{RD} and T_{RD} leads to determination of all the vector spaces H_R^* and V_R^*, respectively, for each constraint $R = 1,\ldots,k$. Then, under a simultaneous application of all the constraints, the dual space at an arbitrary point p of manifold M can be split as

$$T_p^* M = H_S^* \oplus V_T^* , \tag{46}$$

where H_S^* is an ($n-k$)-dimensional and V_T^* is a k-dimensional subspace of $T_p^* M$, defined by

$$H_S^* \equiv \bigcap_{R=1}^{k} H_R^* \quad \text{and} \quad V_T^* \equiv V_1^* \oplus \cdots \oplus V_k^* , \tag{47}$$

respectively. One can also construct the k-dimensional Cartesian product space

$$W^* \equiv W_1^* \times \cdots \times W_k^* \cong \underbrace{\mathbb{R}^* \times \cdots \times \mathbb{R}^*}_{k\ times} .$$

Based on the decomposition (46), any element of $T_p^* M$ can be expressed in the form

$$\underset{\sim}{v}^* = \underset{\sim}{u}_S^* + \underset{\sim}{v}_T^* \quad \text{with} \quad \underset{\sim}{v}_T^* = \underset{\sim}{v}_1^* + \cdots + \underset{\sim}{v}_k^* ,$$

where $\underset{\sim}{u}_S^* \in H_S^*$ and $\underset{\sim}{v}_R^* \in V_R^*$ for $R = 1,\ldots,k$. The above decomposition provides the ground to define composite transformations S_D and T_D between the dual spaces $T_p^* M$ and W^*, as shown in Fig. 2. Specifically, the transformation $S_D : T_p^* M \to W^*$ is defined first by requiring

$$S_D \underset{\sim}{v}^* = S_D \underset{\sim}{v}_T^* = S_D (\underset{\sim}{v}_1^* + \cdots + \underset{\sim}{v}_k^*) \equiv \underset{\sim}{\alpha}^* . \tag{48}$$

Likewise, the composite transformation $T_D : W^* \to V_T^*$ is defined by

$$T_D \underset{\sim}{e}^R = T_{RD} \underset{\sim}{e}^R , \tag{49}$$

so that T_{RD} coincides with the restriction of T_D to W_R^* and

$$T_D \underset{\sim}{\alpha}_R^* = T_D(\alpha_R \underset{\sim}{e}^R) = \alpha_R T_{RD} \underset{\sim}{e}^R = \alpha_R \underset{\sim}{c}^R ,$$

which shows that the image of $\underset{\sim}{\alpha}_R^*$ through T_D lies in $V_R^* \subset V_T^*$ only and not in all of V_T^*.

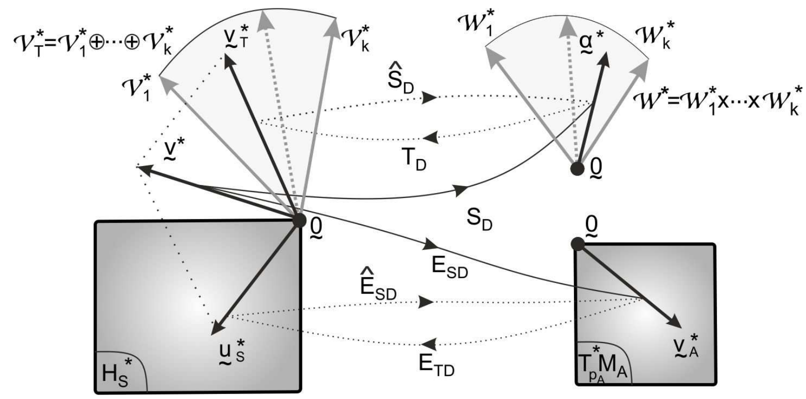

Figure 2: Transformations between the dual spaces of manifolds M , M_A and M_C

Based on the above construction, the properties of S_D are $\eta(S_D) = H_S^*$, $\mathrm{Im}(S_D) = W^*$ and $\rho(S_D) = k$, while the corresponding properties of T_D are $\eta(T_D) = \{\underset{\sim}{0}\}$, $\mathrm{Im}(T_D) = V_T^*$ and $\rho(T_D) = k$. Moreover, by combining Eqs. (48) and (49), it can be shown that

$$S_D T_D \underset{\sim}{\alpha}^* = \underset{\sim}{\alpha}^*, \quad \forall \underset{\sim}{\alpha}^* \in W^* ,$$

which proves that T_D is the right inverse of S_D. In addition, the operator

$$\Pi_D \equiv T_D S_D$$

represents a projection from $T_p^* M$ to V_T^*, with

$$\Pi_D \underset{\sim}{v}^* = \underset{\sim}{v}_T^* \quad \text{and} \quad \underset{\sim}{v}_T^* = \sum_{R=1}^{k} \alpha_R \underset{\sim}{c}^R \in V_T^* .$$

Then, a new bijection $\hat{S}_D$ can be defined, acting from V_T^* to W^* and arising by eliminating the null space H_S^* of S_D through the projection Π_D.

To complete the picture on the dual spaces, an operator E_{SD} can be defined in exactly the same way as in Sect. 3, acting from $T_p^* M$ to $T_{p_A}^* M_A$. In particular, E_{SD} has the properties $\eta(E_{SD}) = V_T^*$, $\mathrm{Im}(E_{SD}) = T_{p_A}^* M_A$ and $\rho(E_{SD}) = m$. Moreover, since E_{SD} is surjective, its right inverse E_{TD} exists and is defined by

$$E_{SD} E_{TD} = I_m , \tag{50}$$

with $\eta(E_{TD}) = \{\underset{\sim}{0}\}$, $\mathrm{Im}(E_{TD}) = H_S^*$, $\rho(E_{TD}) = m$. Then, a projection from $T_p^* M$ to H_S^* results by

$$\Pi_{ED} \equiv E_{TD} E_{SD} .$$

Finally, a companion operator $\hat{E}_{SD}$ to E_{SD} can also be introduced, acting as a bijection from H_S^* to $T_{p_A}^* M_A$, by eliminating the null space of E_{SD} through the projection Π_{ED}.

In closing this section, consider a vector $\underline{u}_T = u^i \underline{e}_i$ of the space H_T, defined by (47). Then,

$$T_R \underline{u}_T = \underline{0}, \qquad R = 1, \dots, k,$$

which in conjunction with (23) leads to

$$\underset{\sim}{a}_i^R u^i = 0. \tag{51}$$

Moreover, by combining (41) and (51) it is concluded that

$$\underset{\sim}{v}_T^*(\underline{u}_T) = 0, \tag{52}$$

which proves that the vector spaces H_T and V_T are orthogonal. By definition, the vectors in H_T represent admissible velocities while the ideal constraint forces, say $\underset{\sim}{f}_{IC}^*$, are covectors produce no work under virtual displacements [8]. This means that they satisfy the condition

$$\underset{\sim}{f}_{IC}^*(\underline{u}_T) = 0. \tag{53}$$

Comparison of (52) and (53) shows that the covectors in V_T^* represent ideal constraint forces.

5 DERIVATION OF EQUATIONS OF MOTION ON THE ORIGINAL MANIFOLD

A central idea of the present work is based on satisfaction of the motion constraints on manifold M by considering the equations of motion on M_A and M_C - or equivalently on M_R, with $R = 1, \dots, k$ - as well. In particular, start by allowing a potential violation of the R-th constraint only and consider the class of Newton covectors on each of the dual spaces W_R^*

$$\underset{\sim}{h}_R^* = \dot{\underset{\sim}{p}}_R^* - \underset{\sim}{f}_R^*. \tag{54}$$

A typical element of W_R^* can be put in the form

$$\underset{\sim}{\alpha}_R^* = \alpha_R \underset{\sim}{e}^R = g_{RR} \dot{q}^R \underset{\sim}{e}^R, \tag{55}$$

where q^R is the coordinate of a point p_R of the space M_R. This space can be generated by a mapping, determined by Eq. (23), from the unique autoparallel of M starting at the solution point p with tangent vector $\underline{c}_R$, as shown in Fig. 3. Moreover, based on conditions (19) and (20), the metric and connection on M_R are selected so that Newton's law holds on them by

$$g_{RR} = c_R^i g_{ij} c_R^j \quad \text{and} \quad \Lambda_{RR}^R = 0. \tag{56}$$

Therefore, taking into account (8) and (55), the first term of $\underset{\sim}{h}_R^*$ in (54) appears in the form

$$\dot{\underset{\sim}{p}}_R^* = (g_{RR} \dot{q}^R)^{\boldsymbol{\cdot}} \underset{\sim}{e}^R. \tag{57}$$

Next, by applying Eq. (42), the second term of $\underset{\sim}{h}_R^*$ is obtained through the transformation

$$\underset{\sim}{f}_R^* = S_{RD} \underset{\sim}{f}_M^* = \overline{f}_R \underset{\sim}{e}^R. \tag{58}$$

The term

$$\overline{f}_R = c_R^i f_i \tag{59}$$

is the component of the force developed when the R-th constraint, represented by Eq. (21), tends to be violated. In general, $f_i = f_i(\underline{q}, \dot{\underline{q}}, t)$. Then, its value at a neighboring point $\hat{p}_R$ is

$$\hat{f}_i = f_i(\underline{q} + s^R \underline{c}_R, \dot{\underline{q}} + \dot{s}^R \underline{c}_R, t).$$

The quantitiy s^R is the canonical coordinate along the autoparallel of M starting at point p with tangent $\underline{c}_R$ [7]. Then, since the values of s^R and $\dot{s}^R$ are infinitesimal

$$\hat{f}_i = f_i(\underline{q},\underline{\dot{q}},t) + \left[\frac{\partial f_i}{\partial q^j}(\underline{q},\underline{\dot{q}},t)c_R^j\right]s^R + \left[\frac{\partial f_i}{\partial \dot{q}^j}(\underline{q},\underline{\dot{q}},t)c_R^j\right]\dot{s}^R + \ldots \tag{60}$$

Next, the choice $s^R = q^R$ is made. Then, combination of equations (58)-(60) and substitution in (54), together with (57), yields the following expression for the Newton covectors on M_R

$$\underset{\sim}{h}_R^* = h_R \underset{\sim}{e}^R = [(\overline{m}_{RR}\dot{q}^R)^{\cdot} + \overline{c}_{RR}\dot{q}^R + \overline{k}_{RR}q^R - \overline{f}_R]\underset{\sim}{e}^R . \tag{61}$$

The new coefficients in the above expression are defined by

$$\overline{m}_{RR} = g_{RR}\,, \quad \overline{c}_{RR} = -c_R^i \frac{\partial f_i}{\partial \dot{q}^j}(\underline{q},\underline{\dot{q}},t)c_R^j \quad \text{and} \quad \overline{k}_{RR} = -c_R^i \frac{\partial f_i}{\partial q^j}(\underline{q},\underline{\dot{q}},t)c_R^j \tag{62}$$

and represent an equivalent mass, damping and stiffness quantity, obtained through a projection along the direction $\underset{\sim}{c}_R$, specified by the R-th constraint.

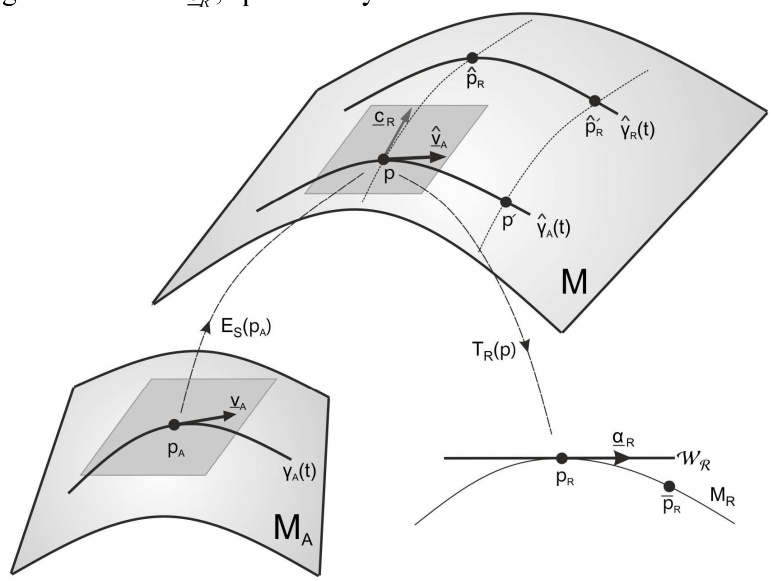

Figure 3: Solution path and deviation along direction of R-th constraint on manifold M

Now, the Newton covectors on manifolds M and M_R are related by

$$\underset{\sim}{h}_R^* = S_{RD}\underset{\sim}{h}_M^* . \tag{63}$$

Next, multiplying both sides of (63) from the left by T_{RD}, projects these equations back to T_p^*M and more specifically to V_R^*. In particular, taking into account (41), this leads to

$$\underset{\sim}{v}_R^* = T_{RD}\underset{\sim}{h}_R^* = \Pi_{RD}\underset{\sim}{h}_M^* . \tag{64}$$

Likewise, multiplying both sides of (18) from the left by E_{TD}, projects these equations back to T_p^*M and more specifically to H_S^* (see Fig. 2) by

$$\underset{\sim}{u}_S^* = E_{TD}\underset{\sim}{h}_A^* = \Pi_{ED}\underset{\sim}{h}_M^* . \tag{65}$$

Combining the above and using the decomposition (46) leads to

$$\underset{\sim}{h}_M^* = \underset{\sim}{h}_S^* + \underset{\sim}{h}_T^* = \Pi_{ED}\underset{\sim}{h}_M^* + \Pi_{TD}\underset{\sim}{h}_M^* , \tag{66}$$

or equivalently

$$\underset{\sim}{h}_M^* = E_{TD}\underset{\sim}{h}_A^* + \sum\nolimits_{R=1}^{k} T_{RD}\underset{\sim}{h}_R^* . \tag{67}$$

However, the equations of motion must always be satisfied on manifold M_A. This means that

$$\underset{\sim}{h}_M^* = \sum_{R=1}^{k} T_{RD} \underset{\sim}{h}_R^* . \tag{68}$$

Moreover, direct application of Eq. (41) gives

$$T_{RD} \underset{\sim}{h}_R^* = T_{RD} (h_R \underset{\sim}{e}^R) = a_i^R h_R \underset{\sim}{e}^i . \tag{69}$$

Finally, adopting traditional notation, the coordinates $\lambda^R \equiv q^R$ can be viewed as Lagrange multipliers, whose action is more general than that obtained in the optimization of functions subjected to constraints. Then, the equations of motion on manifold M take the form

$$(g_{ij}\dot{q}^j)^{\cdot} - \Lambda_{ki}^m g_{mj} \dot{q}^k \dot{q}^j = f_i + \sum_{R=1}^{k} a_i^R [(\overline{m}_{RR} \dot{\lambda}^R)^{\cdot} + \overline{c}_{RR} \dot{\lambda}^R + \overline{k}_{RR} \lambda^R - \overline{f}_R] . \tag{70}$$

For each holonomic constraint, coordinate s^R coincides with the function of constraint $\phi^R(\underline{q})$. Through the choice $q^R = s^R$, this defines a map from M to M_R with explicit form

$$q^R(t) = \phi^R(\underline{q}(t)), \tag{71}$$

where $\underline{q}(t)$ represents the coordinates of point $p(t)$ on M, while q^R is the coordinate of the point $p_R(t)$ mapped on M_R. Then, the tangent map from T_pM to $T_{p_R}M_R$ is also defined by

$$\dot{q}^R(t) = \dot{\phi}^R(\underline{q}(t)), \tag{72}$$

which specifies the corresponding velocity coordinate as well. Therefore, when the R-th constraint is satisfied, then taking into account (21), relations (71) and (72) yield

$$q^R(t) = 0 \quad \text{and} \quad \dot{q}^R(t) = 0 ,$$

respectively, for all times. This implies immediately that

$$\ddot{q}^R(t) = 0 .$$

Consequently, based on Eq. (61), the class of Newton covectors on W_R^* is represented by

$$\underset{\sim}{h}_R^N = -\overline{f}_R \underset{\sim}{e}^R . \tag{73}$$

When the R-th constraint is not satisfied, consider a neighboring point $\overline{p}_R(t)$ of $p_R(t)$ on M_R with coordinate $\overline{q}^R(t)$. If $\underline{\hat{q}}^R(t)$ are the coordinates of the corresponding point $\hat{p}_R(t)$ on M,

$$\overline{q}^R(t) = \phi^R(\underline{\hat{q}}_R(t)) \equiv \phi^R(t) , \tag{74}$$

while the corresponding velocity component is obtained from

$$\dot{\overline{q}}^R(t) = \dot{\phi}^R(t) . \tag{75}$$

Next, consider the motion of point $\overline{p}_R(t)$, defining a curve $\overline{\gamma}_R(t)$ on M_R. The corresponding point $\hat{p}_R(t)$ traces a curve $\hat{\gamma}_R(t)$ on M, lying in the vicinity of $\hat{\gamma}_A(t)$. Finally, it is required that curve $\overline{\gamma}_R(t)$ belongs to the same class of curves with $\gamma_R(t)$, defined by the motion of point $p_R(t)$, so that it leads to the same Newton covector, given by Eq. (73). Consequently, by substituting Eqs. (74) and (75) into (61) and by employing (73), a scalar equation with form

$$(\overline{m}_{RR} \dot{\phi}^R)^{\cdot} + \overline{c}_{RR} \dot{\phi}^R + \overline{k}_{RR} \phi^R = 0 \tag{76}$$

is obtained for each holonomic constraint, which forces both $\dot{\phi}^R$ and ϕ^R to become zero.

When the constraint is nonholonomic, only the following tangent mapping is available

$$\dot{q}^R(t) = \psi^R(\underline{q}(t), \underline{\dot{q}}(t)) .$$

Here, exact satisfaction of the R-th constraint implies that $\psi^R = 0$ for all times. This means that $\dot{\psi}^R = 0$, while the corresponding class of Newton covectors on W_R^* is now represented by

$$\underset{\sim}{h}_R^N = (\bar{k}_{RR} q^R - \bar{f}_R)\underset{\sim}{e}^R \,, \tag{77}$$

instead of (73). Moreover, the perturbation term in the velocity component can be chosen by

$$\dot{\bar{q}}^R(t) = \dot{\psi}^R(\underset{\sim}{\hat{q}}_R(t), \dot{\underset{\sim}{\hat{q}}}_R(t)) \equiv \dot{\psi}^R(t) \,.$$

Consequently, employing Eq. (61) in conjunction with (77) leads to a scalar equation

$$(\bar{m}_{RR}\dot{\psi}^R)^{\cdot} + \bar{c}_{RR}\dot{\psi}^R = 0 \,, \tag{78}$$

which can force only $\dot{\psi}^R$ to become zero.

Conditions (76) and (78) furnish a set of k second order ODEs, since $\bar{m}_{RR}$ is always positive. These equations together with (70) provide a set of $n+k$ second order ODEs in the $n+k$ unknowns q^i and λ^R and must be accompanied by a set of initial conditions. Note that conditions (76) and (78) have a similarity to but are more general than those employed in the so-called Baumgarte stabilization [1, 4]. Here these equations were derived as part of the systematic approach developed and were not introduced artificially. Also, all the coefficients of these equations were determined analytically and not through an adhoc selection.

6 SYNOPSIS

A set of equations of motion for mechanical systems subjected to scleronomic motion constraints was derived. Both holonomic and nonholonomic equality constraints were treated concurrently. The underlying idea was to incorporate the constraints one by one, in a process analogous to that used for setting up the equations of motion, which was equivalent to assigning appropriate inertia and possibly damping and stiffness properties to each constraint equation. This led to a system of second order ODEs for both the generalized coordinates and the Lagrange multipliers, which eliminated the singularities associated with sets of DAEs and the need for an arbitrary selection of terms and parameters required by penalty formulations. Apart from its theoretical elegance, this is expected to have a significant impact in developing efficient methods for the numerical integration of these equations, since no stabilization or scaling is required. In addition, the equations derived include terms activating a mechanism for performing constraint recovery in an automatic manner. Finally, the set of equations obtained can also be extended to systems with rheonomic or unilateral constraints.

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