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INTERIOR POINT METHOD BASED CONTACT ALGORITHM FOR STRUCTURAL ANALYSIS OF ELECTRONIC DEVICE MODELS

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Abstract. In this paper, we present an algorithm for frictionless contact problems of linear elastic bodies with multi-point constraints. Our method is based on interior point method and is developed for large scale stress analysis of electronic device models. These models consist of dozens of parts and contain so many contact constraints and multi-point constraints that make convergence of contact states difficult to achieve. We implemented our algorithm into FrontISTR, which is open-source and large scale finite element structural analysis software, and investigated its performance for a model of laptop PC.

1 INTRODUCTION

In a design process of portable electronic devices such as laptop PCs or mobile phones, it is important to satisfy both of lightness and toughness of the whole body. Recently, the design process has become more difficult than ever, mainly because of two following reasons. The first reason is that components of portable electronic devices such as liquid crystal display and printed circuit board have become thin and small. Therefore, each component does not have sufficient stiffness. The second reason is increase of portability. Portable electronic devices encounter various kinds of external loads, for example pressure in bag or drop impact. Today, structural analysis based on finite element method is widely used in order to improve the design quality and reduce the cost of experimental production. Recent finite element models are usually built by assembled parts to evaluate the stiffness of the whole body. This kind of problems are formulated as static stress analysis with contact constraints and multi-point constraints between various parts.

It should be noted that electronic devices consist of many thin parts such as liquid crystal display, printed circuit board and covers, and other parts are arranged between them. Contacts occur everywhere inside electronic devices when external loads are applied to them. Thus, the finite element models have not only large number of degrees of freedom (DOFs) but also large number of contact constraints.

Major algorithms to solve this kind of contact problems are introduced in Wriggers[1]. As

the major algorithm, active set method is well known as a simple and widely used method. In this method, a candidate contact node is called active when it is in contact and active set is the set of all active candidate contact nodes. Active set is given from an initial configuration of model and then iterative update of active set is done until the complementary conditions between the gap and contact force converge. In each iteration, trial deformation analysis in which current active contact nodes are constrained is done and then active set is updated by the results. It is also called as trial-and-error method. Applying active set method to structural analysis of electronic devices leads to large number of iteration to determine the contact state. Since a matrix solver routine is called once or more times per iteration in computation, increase of the number of iteration directly affects the total computational time.

We focus on interior point methods [2] as an efficient strategy for this kind of problems. Interior point methods are said to be efficient algorithms to solve convex nonlinear problems with the large number of constraints. Christensen *et al*.[3] applied primal-dual interior point method to linear elastic contact problem with friction. They also compared interior point method with semismooth Newton method, which is a direct expansion of Newton method to non-differencial functions, and concluded that semismooth Newton method is faster and robust. However, their models have quite small dofs and they also indicated that interior point method might need less iterations than semismooth Newton method for problems with a huge number of potential contact nodes. Tanoh *et al*.[4] also applied primal and primal-dual interior point method, and showed that interior point methods are efficient for large scale problems. Miyamura *et al*.[5] proposed the combination of active set method and primal interior point method and showed that their algorithm are faster than simple application of interior point methods in some examples.

In this paper, we propose an interior point method based algorithm to solve the contact problem that contains many contact constraints and multi-point constraints. In our method, multi-point constraints are removed by quadratic penalty method and then primal-dual interior point method is applied to the problem. Furthermore, we implement our algorithm into FrontISTR[6], which is open-source and large scale finite element structural analysis software, and investigate its performance by using laptop PC model.

This paper is organized as follows: Formulation of contact problem is briefly explained in section 2. In section 3, primal-dual algorithm for contact problem given by Tanoh *et al*.[4] is introduced and our algorithm is proposed. Numerical results are given in section4, and conclusion is presented in section 5.

2 FORMULATION OF CONTACT PROBLEM

In this paper, we consider three-dimensional, small deformation and frictionless contact problem of elastic bodies that are discretized by finite element method. We use node-to-segment discretization for contact model. It is because mesh of contact surfaces is non-conforming since the mesh is generated in a part-by-part manner for electronic device models.

Let n denote the number of nodes and n_c denote the number of contact candidate nodes. The nodal displacement vector, external load vector and stiffness matrix are represented by $\boldsymbol{u} \in \mathbb{R}^{3n}$, $\boldsymbol{f} \in \mathbb{R}^{3n}$ and $\boldsymbol{K} \in \mathbb{R}^{3n \times 3n}$, respectively. Let r_i $(i = 1, 2, \dots, n_c)$ denote the contact force for *i*-th contact constraint. The equibrium equations are given by

$$\boldsymbol{K}\boldsymbol{u} = \boldsymbol{f} + \sum_{i=1}^{n_c} \boldsymbol{T}_i \, r_i, \tag{1}$$

where $\boldsymbol{T}_i \in \mathbb{R}^{3n}$ is the vectors that transform contact force r_i to that of global coordinate. For the node-to-segment contact model, the components of $\boldsymbol{T}_i$ is calculated from the normal vector and the shape function at a projection point of a slave node to a master segment. A slave node and a master segment are illustrated in fig. 1.

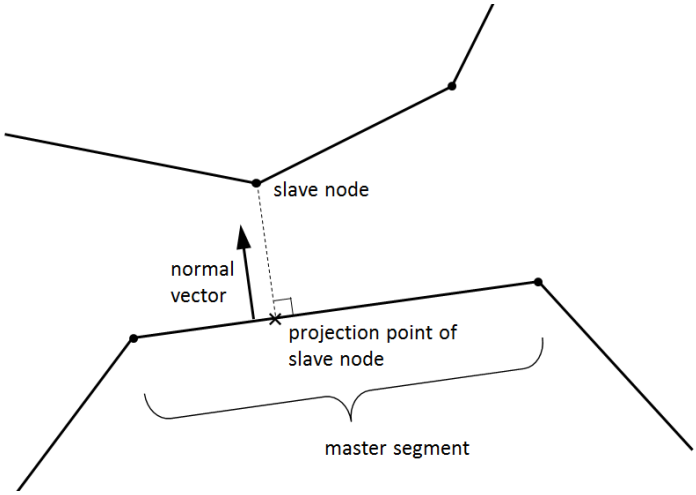

Figure 1: slave node and master segment.

Furthermore, Let h_i $(i = 1, 2, \dots, n_c)$ denote initial gap of i-th contact constraint, then the gap at current configulation can be represented by $\boldsymbol{T}_i^{\mathrm{T}}\boldsymbol{u} + h_i$. For the gap and contact force, the following conditions must be satisfied:

$$r_i\left(\boldsymbol{T}_i^{\mathrm{T}}\boldsymbol{u} + h_i\right) = 0, (i = 1,2, \dots, n_c), \tag{2}$$

$$r_i \geq 0\,, (i = 1,2, \dots, n_c), \tag{3}$$

$$\boldsymbol{T}_i^{\mathrm{T}}\boldsymbol{u} + h_i \geq 0, (i = 1,2, \dots, n_c). \tag{4}$$

A contact candidate node is said to be “free” when the gap is positve and the contact force is zero, and is said to be “in contact” when its gap is zero and contact force is positive. Equations (2), (3) and (4) represent that the state of each contact constraint is either free or in contact.

We also point out that eqs.(1), (2), (3) and (4) are the Karush Kuhn Tucker (KKT) condition of the minimization problem of the following objective function under the constraint if eq.(4):

$$I(\boldsymbol{u}) \equiv \frac{1}{2}\boldsymbol{u}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{u} - \boldsymbol{f}^{\mathrm{T}}\boldsymbol{u}. \tag{5}$$

Thus, it is possible to apply various optimization methods to contact problems.

3 INTERIOR POINT METHODS FOR CONTACT PROBLEM

In this section, we describe primal-dual interior point algorithm for contact problem given by Tanoh *et al.*[4], and then represent proposed algorithm.

3.1 Primal-dual interior point algorithm for contact problem

The main idea of interior point method is to remove inequality constraints by using a logarithmic barrier function. We consider the following barrier problem:

$$\min_{\boldsymbol{u}} I(\boldsymbol{u}) - \mu \sum_{i=1}^{n_c} \log\left(\boldsymbol{T}_i^{\mathrm{T}}\boldsymbol{u} + h_i\right), \tag{6}$$

where μ is a positive real parameter. The second barrier term in eq.(6) diverges at the boundary of inequality (4), so adding this term makes the optimal point of original problem move inside the constraind region. μ is the coefficient of the barrier term and it is known that the optimal point of barrier problem converges to that of original problem when μ tends to zero. The orbit that optimal point makes when μ moves from infinity to zero is called central path and interior point methods follow this path. The basic procedure is as follows: First, initial internal point is set in some way. Second, Newton direction of barrier problem is calculated and the step is taken in the direction keeping the updated point inside the constrained region. Third, μ is decreased in some way. Second and third procedures are repeated until current point converges to KKT point of original problem.

We can straightforwardly get the first order optimal condition of barrier problem in eq.(6) by differentiating object function of eq.(6):

$$\boldsymbol{K}\boldsymbol{u} - \boldsymbol{f} - \mu \sum_{i=1}^{n_c} \frac{\boldsymbol{T}_i}{(\boldsymbol{T}_i^{\mathrm{T}}\boldsymbol{u} + h_i)} = \boldsymbol{0}. \tag{7}$$

Let $r_i = \mu/\left(\boldsymbol{T}_i^{\mathrm{T}}\boldsymbol{u} + h_i\right)$, then we can see that eqs.(7) and (4) are equivalent to following eqs.(8), (9), (3) and (4):

$$\boldsymbol{K}\boldsymbol{u} - \boldsymbol{f} - \sum_{i=1}^{n_c} \boldsymbol{T}_i r_i = \boldsymbol{0}, \tag{8}$$

$$r_i\left(\boldsymbol{T}_i^{\mathrm{T}}\boldsymbol{u} + h_i\right) = \mu, (i = 1,2, \dots, n_c). \tag{9}$$

We can get this condition just by replacing the right hand side of eq.(2) with μ.

In primal-dual algorithm, both $\boldsymbol{u}$ and r_i are treated as unknown variables. Let $\Delta\boldsymbol{u}$ and Δr_idenote the Newton direction of eqs.(8) and (9), respectively, then the linear equations that $\Delta\boldsymbol{u}$ and Δr_i should satisfy are as follows:

$$\boldsymbol{K}\Delta\boldsymbol{u} - \sum_{i=1}^{n_c} \boldsymbol{T}_i \Delta r_i = -\left(\boldsymbol{K}\boldsymbol{u} - \boldsymbol{f} - \sum_{i=1}^{n_c} \boldsymbol{T}_i r_i\right), \tag{10}$$

$$r_i\boldsymbol{T}_i^{\mathrm{T}}\Delta\boldsymbol{u} + \Delta r_i\left(\boldsymbol{T}_i^{\mathrm{T}}\boldsymbol{u} + h_i\right) = \mu - r_i\left(\boldsymbol{T}_i^{\mathrm{T}}\boldsymbol{u} + h_i\right), (i = 1,2, \dots, n_c). \tag{11}$$

The coefficient matrices of eqs.(10) and (11) are not symmetric, so we have to use direct

solvers or unsymmetric iterative solvers such as GMRES. We can get positive definite symmetric coefficient matrix by removing Δr_i from eq.(10). The resulting equation is

$$\left(\boldsymbol{K} + \sum_{i=1}^{n_c} \frac{r_i}{\boldsymbol{T}_i^{\mathrm{T}}\boldsymbol{u} + h_i} \boldsymbol{T}_i \boldsymbol{T}_i^{\mathrm{T}} \right) \Delta \boldsymbol{u} = - \left(\boldsymbol{K}\boldsymbol{u} - \boldsymbol{f} - \sum_{i=1}^{n_c} \frac{\mu}{\boldsymbol{T}_i^{\mathrm{T}}\boldsymbol{u} + h_i} \boldsymbol{T}_i \right). \tag{12}$$

After the calculation of $\Delta \boldsymbol{u}$ from eq.(12), Δr_i is given by:

$$\Delta r_i = \frac{\mu - r_i \boldsymbol{T}_i^{\mathrm{T}} \Delta \boldsymbol{u}}{\boldsymbol{T}_i^{\mathrm{T}}\boldsymbol{u} + h_i} - r_i, (i = 1,2, \dots, n_c). \tag{13}$$

The residual of eqs.(8) and (9) is defined as

$$\phi(\boldsymbol{u}, \boldsymbol{r}, \mu) = \left(\left\| \boldsymbol{K}\boldsymbol{u} - \boldsymbol{f} - \sum_{i=1}^{n_c} \boldsymbol{T}_i r_i \right\|_2^2 + \sum_{i=1}^{n_c} \left| r_i \left(\boldsymbol{T}_i^{\mathrm{T}}\boldsymbol{u} + h_i \right) - \mu \right|^2 \right)^{\frac{1}{2}}, \tag{14}$$

where $\boldsymbol{r} = \left(r_1, r_2, \dots, r_{n_c} \right)^{\mathrm{T}}$. The primal-dual interior point algorithm is depicted in fig. 2. In this algorithm, matrix solver routine is called once per iteration of the main loop in computation.

Set $\mu > 0, \epsilon > 0, \eta > 0, \tau \in (0,1)$.
Choose initial $\boldsymbol{u}, \boldsymbol{r}$ from interior point of (3) and (4)

Main loop
If $\phi(\boldsymbol{u}, \boldsymbol{r}, \mu) < \epsilon$ **then stop**
Get $\Delta \boldsymbol{u}, \Delta \boldsymbol{r}$ from (12) and (13), where $\Delta \boldsymbol{r} = \left(\Delta r_1, \Delta r_2, \dots, \Delta r_{n_c} \right)$
Use backtracking to find $\sigma \in (0,1]$ that satisfies following conditions:

$$\phi(\boldsymbol{u} + \sigma \Delta \boldsymbol{u}, \boldsymbol{r} + \sigma \Delta \boldsymbol{r}, \mu) - \phi(\boldsymbol{u}, \boldsymbol{r}, \mu) < \tau \sigma \nabla \phi(\boldsymbol{u}, \boldsymbol{r}, \mu) \text{ (Armijo rule)},$$
$$\boldsymbol{T}_i^{\mathrm{T}}(\boldsymbol{u} + \sigma \Delta \boldsymbol{u} + h_i) > 0 \ (i = 1,2, \dots, n_c),$$
$$r_i + \sigma \Delta r_i > 0 \ \ (i = 1,2, \dots, n_c).$$

Update $\boldsymbol{u} \leftarrow \boldsymbol{u} + \sigma \Delta \boldsymbol{u}, \boldsymbol{r} \leftarrow \boldsymbol{r} + \sigma \Delta \boldsymbol{r}$
Decrease μ as follows: $\mu \leftarrow \eta \mu$
Continue loop.

Figure 2: primal-dual algorithm

3.2 Proposed algorithm

We describe proposed algorithm here. Multi-point constraints are considered in the algorithm and some minor modifications are applied to primal-dual interior point algorithm from the test by electronic device models.

In this paper, we consider the multi-point constraints that tie two surfaces. Each slave node is forced to coincide with its projectioin point on master surface. In this case, the multi-point constraints are presented as the following linear equations:

$$\boldsymbol{B}_i^T \boldsymbol{u} = \boldsymbol{0}, (i = 1,2, \dots, n_M) \tag{15}$$

where n_M is the number of multi-point constraints and $\boldsymbol{B_i}$ is coefficient vector.

We use quadratic penalty method and remove these multi-point constraints. Then, objective function is modified as follows:

$$I_M(\boldsymbol{u}) \equiv \frac{1}{2}\boldsymbol{u}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{u} - \boldsymbol{f}^{\mathrm{T}}\boldsymbol{u} + \frac{1}{2}c\sum_{i=1}^{n_M}\boldsymbol{u}\boldsymbol{B}_i\ \boldsymbol{B}_i^T\boldsymbol{u}. \tag{16}$$

We apply primal-dual interior algorithm to I_M. Linear equation (12) to calculate $\Delta\boldsymbol{u}$ is modified as follows:

$$\left(\boldsymbol{K} + \sum_{i=1}^{n_c}\frac{r_i}{\boldsymbol{T}_i^{\mathrm{T}}\boldsymbol{u} + h_i}\boldsymbol{T}_i\,\boldsymbol{T}_i^{\mathrm{T}} + c\sum_{i=1}^{n_M}\boldsymbol{B}_i\ \boldsymbol{B}_i^T\right)\Delta\boldsymbol{u} = -\left(\boldsymbol{K}\boldsymbol{u} - \boldsymbol{f} - \sum_{i=1}^{n_c}\frac{\mu}{\boldsymbol{T}_i^{\mathrm{T}}\boldsymbol{u} + h_i}\boldsymbol{T}_i + c\sum_{i=1}^{n_M}\boldsymbol{B}_i\ \boldsymbol{B}_i^T\boldsymbol{u}\right). \tag{17}$$

Δr_i can be calculated by (13). The residual of this system is defined as

$$\phi_M(\boldsymbol{u},\boldsymbol{r},\mu) = \left(\left\|\boldsymbol{K}\boldsymbol{u} - \boldsymbol{f} - \sum_{i=1}^{n_c}\boldsymbol{T}_i\,r_i + c\sum_{i=1}^{n_M}\boldsymbol{B}_i\ \boldsymbol{B}_i^T\boldsymbol{u}\right\|_2^2 + \sum_{i=1}^{n_c}\left|r_i\left(\boldsymbol{T}_i^{\mathrm{T}}\boldsymbol{u} + h_i\right) - \mu\right|^2\right)^{\frac{1}{2}}. \tag{18}$$

Then we show the present algorithm in fig. 3. The present algorithm differs in two points from original primal-dual interior point algorithm. First, we adopt different step factors σ_1 and σ_2 for $\boldsymbol{u}$ and $\boldsymbol{r}$, respectively. It implies that an interior point method for nonlinear functions is applied to the quadratic function I_M. In structural analysis of electronic device models, compared with the same step factors, the different step factors can accelerate iteration convergence. Next, we introduce 'decrease ratio' of μ that is equivalent to the interior point method for quadratic programming. The primal-dual algorithm in Tanoh *et al.*[4] decreases the value of μ regardless of the step length, so it tends to decrease μ excessively.

Set $\mu > 0, \epsilon > 0, \eta > 0, \tau \in (0,1)$.
Choose initial $\boldsymbol{u}, \boldsymbol{r}$ from interior point of (3) and (4)

Main loop
If $\phi_M(\boldsymbol{u},\boldsymbol{r},\mu) < \epsilon$ **then stop**
Get $\Delta\boldsymbol{u}, \Delta\boldsymbol{r}$ from (17) and (13)
Use backtracking to find $\sigma_1, \sigma_2 \in (0,1]$ that satisfies following conditions:

$$\phi(\boldsymbol{u} + \sigma_1\Delta\boldsymbol{u}, \boldsymbol{r} + \sigma_2\Delta\boldsymbol{r}, \mu) - \phi(\boldsymbol{u},\boldsymbol{r},\mu) < \tau\sigma_1\nabla\phi(\boldsymbol{u},\boldsymbol{r},\mu) \text{ (Armijo rule)},$$
$$\boldsymbol{T}_i^{\mathrm{T}}(\boldsymbol{u} + \sigma_1\Delta\boldsymbol{u} + h_i) > 0\ (i = 1,2,\dots,n_c),$$
$$r_i + \sigma_2\Delta r_i > 0\ \ (i = 1,2,\dots,n_c).$$

Update $\boldsymbol{u} \leftarrow \boldsymbol{u} + \sigma_1\Delta\boldsymbol{u}, \boldsymbol{r} \leftarrow \boldsymbol{r} + \sigma_2\Delta\boldsymbol{r}$
Decrease μ as follows: $\mu \leftarrow (\sigma\eta + (1-\sigma))\mu$, where $\sigma = (\sigma_1 + \sigma_2)/2$
Continue loop.

Figure 3: proposed algorithm

4 NUMERICAL EXPERIMENTS

In this section, we present the results of numerical experiment using a laptop PC model. We implemented proposed algorithm into FrontISTR[6]. Since FrontISTR supports contact analysis function by an active set method and an augmented Lagrange method, we could easily implement our algorithm by using its data structure in the contact analysis module.

Figure 4 shows a laptop PC model which we use for this numerical experiment. This analysis aims to evaluate stress at the liquid crystal display when the rear cover is pressed by an external force. The laptop PC model consists of 25 parts and its the total number of DOFs is 523,426. Contacts or multi-point constraints are defined between these parts and the total numbers of them are 50,658 and 2,492, respectively.

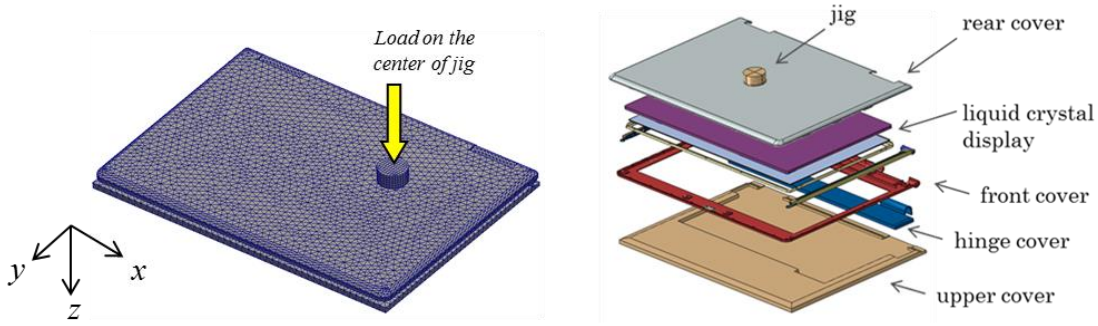

Figure 4: Laptop PC model. Left : Mesh model, Right : Illustrated parts breakdown of laptop PC.

We used the following parameters: initial $\mu = 0.5$, $\epsilon = 10^{-6}$, $\eta = 0.3, 0.4, 0.5, 0.6, 0.7$. As for the contact conditions of which initial gap is to be zero, we set $h_i = 10^{-8}$ so that an initial condition of $\boldsymbol{u} = \boldsymbol{0}$ is within the interior point.

The number of iterations to converge required by the present algorithm and an active set method is shown in Table 1. A penalty method is used in an active set method. In this problem, an active set method needs one matrix solver execution per iteration of the main loop since the objective function is quadratic. Thus, this iteration number is the same as the total number of matrix solver executions for both methods. The maximum values of the nodal displacements are almost the same for two methods. The iteration number of the present algorithm is less than that of an active set method except the case of $\eta = 0.3$. The numerical result of $\eta = 0.3$ implies that the convergence of the present algorithm become worse if we use too small η, that is, decrease μ too excessively.

Table 1: Iteration number of active set method and interior point method for the laptop PC model.

methods	η	Max. displacement of z direction [mm]	iteration
Active set method		4.234	60
Proposed algorithm	0.3	4.235	67
	0.4	4.235	55
	0.5	4.235	48
	0.6	4.235	51
	0.7	4.235	58

5 CONCLUSION

We proposed an interior point method based contact algorithm. It is intended to solve large scale contact problems with multi-point constraints which derive from stress analysis of electronic device models. We also implemented this algorithm into the open-source

FrontISTR, and investigated the performance by the contact analysis of a laptop PC model. From the numerical result, we have confirmed the fast convergence of our algorithm but it strongly depends on the decrease ratio of parameter μ. Therefore, more efficient and robust updating strategy of μ remains to be open.

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