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TWO LEVEL FETI METHOD FOR TRANSIENT PROBLEMS

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Abstract. In this paper we deal with a variant of FETI (Finite Element Tearing and Interconnecting) domain decomposition method for transient problems of linear elasticity. The basic idea of this method is based on the standard FETI-DP approach, but the implementation is more close to classical FETI/TFETI method. This method can be viewed as two level FETI method. The effective stiffness matrices are assembled for floating subdomains and the continuity of the solution across the interfaces is enforced by two sets of Lagrange multipliers. The first set enforces the continuity at the "corner" nodes. The continuity on the rest of the interface is obtained within the iteration process as in standard approach. The behavior of the proposed method is demonstrated on academic benchmark implemented within the MatSol library.

1 INTRODUCTION

The goal of this paper is to describe a variant of FETI method based on our variant of the FETI type domain decomposition method called Total FETI [3]. The original FETI method, also called FETI-1 method, was introduced by Farhat and Roux [5]. In both methods a body is decomposed into several non-overlapping subdomains and the continuity between the subdomains is prescribed by gluing conditions and enforced by Lagrange multipliers. Using a theory of duality, a smaller and relatively well conditioned dual problem can be derived and efficiently solved by suitable variant of the conjugate gradient algorithm. In Total FETI method [3] also the Dirichlet boundary conditions are enforced by Lagrange multipliers.

Another variant of FETI is the FETI-DP method, where the continuity of the solution along the subdomain interfaces is enforced by Lagrange multipliers except for the subdomain corners, which remain primal variables. The FETI-DP method was originally introduced by Farhat, Lesoinne, Le Tallec, Pierson, and Rixen [6] and analyzed for scalar two dimensional problems by Mandel and Tezaur [7].

Here we describe Total FETI-DP method, which can be viewed as 2 level Total FETI. As for TFETI a body is decomposed into several non-overlapping subdomains and the continuity between the subdomains and the Dirichlet boundary condition are enforced by two sets of Lagrange multipliers. The first set enforces the continuity at the "corner" nodes. The continuity on the rest of the interface is obtained within the iteration process as in standard approach.

The paper is organized as follows. After introducing a transient problem of linear elasticity and its domain decomposition in Section 2, we describe discretization for space and time domains in Section 3. In Section 4 we present Total FETI-DP method. The results of numerical experiments are illustrated in Section 5.

2 TRANSIENT PROBLEM AND DOMAIN DECOMPOSITION

Let us consider homogeneous isotropic elastic body, which occupies, in a reference configuration, a domain Ω in $\mathbb{R}^d$, $d = 2, 3$, with a sufficiently smooth boundary Γ. Suppose that Γ consists of two disjoint parts Γ_U and Γ_F, $\Gamma = \overline{\Gamma}_U \cup \overline{\Gamma}_F$, and consider zero displacements on Γ_U and given forces $\mathbf{F} : \Gamma_F \to \mathbb{R}^d$ on Γ_F, see Figure 1. We admit $\Gamma_U = \emptyset$, but we assume for simplicity that Γ is sufficiently smooth so that for almost every $\mathbf{x} \in \Gamma$, there is a unique external normal $\mathbf{n} = \mathbf{n}(\mathbf{x})$.

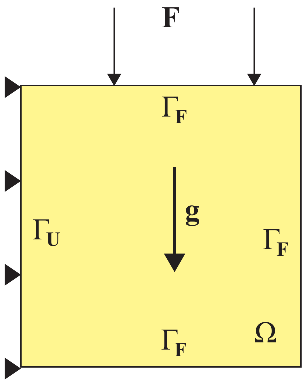

Figure 1: Model problem

The mechanical properties of Ω are defined by the Young modulus E, the Poisson ratio ν, and the density ρ. The Young modulus and the Poisson ratio define the entries of the elasticity tensor $c_{ijkl} : \Omega \to \mathbb{R}^d$, while the density defines the inertia forces and the vector of body forces $\mathbf{g} : \Omega \to \mathbb{R}^d$. Since we assume that the body is homogeneous, the mechanical properties of the body are independent of $\mathbf{x}$.

Under the assumption of linear elasticity and using the summation convention, the stress tensor $\sigma(\mathbf{u})$ is given by

$$\sigma_{ij}(\mathbf{u}) = c_{ijkl} e_{kl}(\mathbf{u}),$$

where

$$e_{ij}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{1}$$

The conditions that should be satisfied by the displacements $\mathbf{u} : \Omega \times \langle 0, T\rangle \to \mathbb{R}^d$, $T > 0$ are

$$\rho\ddot{\mathbf{u}} - \operatorname{div}\sigma(\mathbf{u}) = \mathbf{g} \quad \text{in } \ \Omega \ \times \langle 0, T\rangle, \tag{2}$$

$$\mathbf{u} = \mathbf{o} \quad \text{on } \Gamma_U \times \langle 0, T\rangle, \tag{3}$$

$$\sigma(\mathbf{u})\,\mathbf{n} = \mathbf{F} \quad \text{on } \Gamma_F \times \langle 0, T\rangle, \tag{4}$$

$$\mathbf{u}(.,0) = \mathbf{u}_0 \ \text{in } \ \Omega, \tag{5}$$

$$\dot{\mathbf{u}}(.,0) = \dot{\mathbf{u}}_0 \ \text{in } \ \Omega, \tag{6}$$

with the Newton equation of motion (2), the classical boundary conditions (3) and (4), and the initial values (5) and (6).

Let $\mathcal{V} = \left\{\mathbf{v} \in (H^1(\Omega))^d : \ \mathbf{v} = \mathbf{o} \text{ on } \Gamma_U\right\}$ be a space with the test functions. Then the weak formulation is: for almost every time τ, find $\mathbf{u}(\,\cdot\,,\tau) \in \mathcal{V}$ such that

$$m(\ddot{\mathbf{u}}, \mathbf{v}) + a(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \qquad \mathbf{v} \in \mathcal{V}, \tag{7}$$

$$(\mathbf{u}(\cdot, 0), \mathbf{v}) = (\mathbf{u}_0, \mathbf{v}), \qquad \mathbf{v} \in \mathcal{V}, \tag{8}$$

$$(\dot{\mathbf{u}}(\cdot, 0), \mathbf{v}) = (\dot{\mathbf{u}}_0, \mathbf{v}), \qquad \mathbf{v} \in \mathcal{V}, \tag{9}$$

with the definitions

$$m(\ddot{\mathbf{u}}, \mathbf{v}) = \int_\Omega \varrho \ddot{\mathbf{u}} \cdot \mathbf{v}\, \mathrm{d}\Omega,$$

$$a(\mathbf{u}, \mathbf{v}) = \int_\Omega c_{ijkl} e_{ij}(\mathbf{u})\, e_{kl}(\mathbf{v})\, \mathrm{d}\Omega,$$

$$(\mathbf{f}, \mathbf{v}) = \int_\Omega \mathbf{g} \cdot \mathbf{v}\, \mathrm{d}\Omega + \int_{\Gamma_F} \mathbf{F} \cdot \mathbf{v}\, \mathrm{d}\Gamma.$$

To apply the TFETI domain decomposition, we tear the body from the part of the boundary with the Dirichlet boundary condition, decompose the domain Ω into s subdomains Ω^p, assign each subdomain a unique number, and introduce new "gluing" conditions on the artificial intersubdomain boundaries and on the boundaries with imposed Dirichlet conditions. For the artificial intersubdomain boundaries, we introduce this notation. Γ_G^{pq} denotes the part of Γ that is glued to Ω^q and Γ_G denotes the part of Γ that is glued to the other subdomains. Obviously $\Gamma_G^{pq} = \Gamma_G^{qp}$. An auxiliary decomposition of the problem of Figure 1 with artificial intersubdomain boundaries is in Figure 2. The gluing conditions require continuity of the displacements and of their normal derivatives across the intersubdomain boundaries. The procedure is the same as that for the static problem [4].

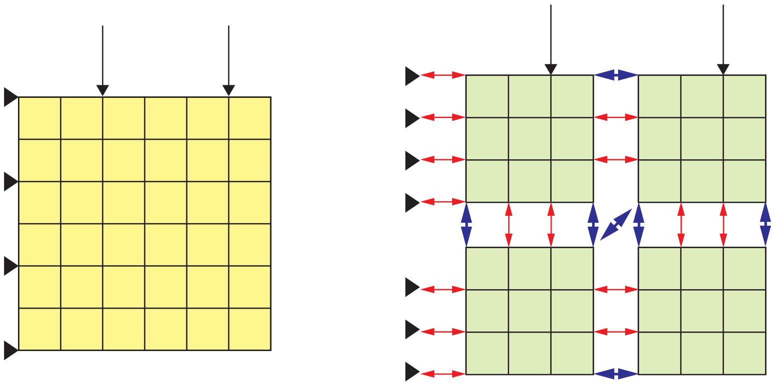

Figure 2: Domain decomposition

3 DISCRETIZATION

We use the finite element discretization in space to get the matrix counterpart of (7)–(9) at the time τ

$$\begin{aligned} \mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} &= \mathbf{f} - \mathbf{B}^T\boldsymbol{\lambda}, &(10)\\ \mathbf{B}\mathbf{u} &= \mathbf{c} = \mathbf{o}, &(11)\\ \boldsymbol{\lambda}^T(\mathbf{B}\mathbf{u} - \mathbf{c}) &= 0. &(12)\end{aligned}$$

Due to the TFETI domain decomposition, the finite element semi-discretization in space of the domain Ω, results in the block diagonal stiffness matrix $\mathbf{K} = \mathrm{diag}\,(\mathbf{K}_1, \ldots, \mathbf{K}_s)$ of the order n with the sparse positive semidefinite diagonal blocks $\mathbf{K}_p$ that correspond to the subdomains Ω^p. The same structure has a positive definite mass matrix $\mathbf{M} = \mathrm{diag}\,(\mathbf{M}_1, \ldots, \mathbf{M}_s)$. The decomposition induces also the block structure of the vector of nodal forces $\mathbf{f} = \mathbf{f}_\tau \in \mathbb{R}^n$ at time τ and the vector of nodal displacements $\mathbf{u} = \mathbf{u}_\tau \in \mathbb{R}^n$ at time τ.

The matrix $\mathbf{B} \in \mathbb{R}^{m\times n}$ with the rows $\mathbf{b}_i$ and the vector $\mathbf{c} \in \mathbb{R}^m$ with the entries $c_i = 0$ enforce the prescribed zero displacements on the part of the boundary with imposed Dirichlet condition and the continuity of the displacements across the auxiliary interfaces. The continuity requires that $\mathbf{b}_i\mathbf{u} = c_i = 0$, where $\mathbf{b}_i$ are vectors of the order n with zero entries except 1 and -1 at appropriate positions. In our implementation we assembled the matrix $\mathbf{B}$ in the orthogonal form. Typically m is much smaller than n. Finally, $\boldsymbol{\lambda} \in \mathbb{R}^m$ denote the vector of Lagrange multipliers $\boldsymbol{\lambda} = \boldsymbol{\lambda}_\tau \in \mathbb{R}^m$, at time τ.

For the time discretization, we use the Newmark scheme [8] with the regular partition of the time interval $\langle 0, T\rangle$ into n_T steps:

$$0 = \tau_0 < \tau_1 \ldots < \tau_{n_T} = T, \quad \tau_k = k\Delta, \quad \Delta = T/n_T, \quad k = 0, \ldots, n_T.$$

Algorithm 1 *(Newmark scheme.)*
Step 1. $\{$*Initialization*$\}$
Set the initial conditions in time $\tau = 0$ *for* $\mathbf{u}, \dot{\mathbf{u}}, \ddot{\mathbf{u}}$. *Set* Δ *and parameter* β *and compute the constants:*

$$a_0 = \frac{1}{\beta \Delta^2}, \quad a_2 = \frac{1}{\beta \Delta}, \quad a_3 = \left(\frac{1}{2\beta} - 1 \right)$$

Step 2. $\{$*Effective stiffness matrix assembly*$\}$

$$\mathbf{K}^{ef} = \mathbf{K} + a_0 \mathbf{M}$$

Step 3. **for** $\tau = \tau_k, \quad k = 0, \ldots, n_T$
$\{$*Right hand side*$\}$

$$\mathbf{f}^{ef} = \mathbf{f}_{\tau+\Delta} + \mathbf{M}(a_0 \mathbf{u}_\tau + a_2 \dot{\mathbf{u}}_\tau + a_3 \ddot{\mathbf{u}}_\tau)$$

$\{$*Solution to the minimization problem*$\}$

$$\min \frac{1}{2} \mathbf{u}^\top \mathbf{K}^{ef} \mathbf{u} - \mathbf{u}^\top \mathbf{f}^{ef} \quad \textit{subject to} \quad \mathbf{B}\mathbf{u} = \mathbf{c} \tag{13}$$

end for

4 TOTAL FETI-DP

The problem (13) has the same structure as in standard Total FETI method and could be solved by this standard approach. However, to describe the TFETI-DP method, we will consider the problem (13) in the form

$$\min \frac{1}{2} \mathbf{u}^\top \mathbf{K}^{ef} \mathbf{u} - \mathbf{u}^\top \mathbf{f}^{ef} \quad \text{subject to} \quad \left\{ \begin{array}{l} \mathbf{B}_0 \mathbf{u} = \mathbf{c}_0 \\ \mathbf{B}_1 \mathbf{u} = \mathbf{c}_1 \end{array} \right. , \tag{14}$$

where the equality constraints are split up into two parts. The first part $\mathbf{B}_0 \mathbf{u} = \mathbf{c}_0 := \mathbf{o}$ consists of m_0 equalities enforcing the continuity in the subdomain corner nodes of each subdomain (see Figure 2, blue arrows), while $\mathbf{B}_1 \mathbf{u} = \mathbf{c}_1$ consists of m_1 equalities enforcing the continuity across the rest of the subdomain interfaces and the Dirichlet condition (see Figure 2, red arrows).

The KKT optimality conditions lead to the saddle point problem

$$\left[\begin{array}{cc|c} \mathbf{K}^{ef} & \mathbf{B}_0^\top & \mathbf{B}_1^\top \\ \mathbf{B}_0 & \mathbf{O} & \mathbf{O} \\ \hline \mathbf{B}_1 & \mathbf{O} & \mathbf{O} \end{array} \right] \left[\begin{array}{c} \mathbf{u} \\ \boldsymbol{\lambda}_0 \\ \hline \boldsymbol{\lambda}_1 \end{array} \right] = \left[\begin{array}{c} \mathbf{f}^{ef} \\ \mathbf{c}_0 \\ \hline \mathbf{c}_1 \end{array} \right] \tag{15}$$

or

$$\left[\begin{array}{c|c} \tilde{\mathbf{K}}^{ef} & \tilde{\mathbf{B}}^\top \\ \hline \tilde{\mathbf{B}} & \mathbf{O} \end{array} \right] \left[\begin{array}{c} \tilde{\mathbf{u}} \\ \hline \tilde{\boldsymbol{\lambda}} \end{array} \right] = \left[\begin{array}{c} \tilde{\mathbf{f}}^{ef} \\ \hline \tilde{\mathbf{c}} \end{array} \right], \tag{16}$$

where $\tilde{\mathbf{K}}^{ef}, \tilde{\mathbf{B}}, \tilde{\mathbf{u}}, \tilde{\boldsymbol{\lambda}}, \tilde{\mathbf{f}}^{ef}$, and $\tilde{\mathbf{c}}$ respect the block structure indicated in (15).

In order to eliminate the primal variables $\tilde{u}$ from the system (16) we express

$$\tilde{\mathbf{u}} = (\tilde{\mathbf{K}}^{ef})^{-1}(\tilde{\mathbf{f}}^{ef} - \tilde{\mathbf{B}}^\top \tilde{\boldsymbol{\lambda}}) \tag{17}$$

from the first equation. Substituting (17) into the second equation of (16) and using the lumped preconditioner $\mathbf{M}_L^{-1} = \tilde{\mathbf{B}}\tilde{\mathbf{K}}^{ef}\tilde{\mathbf{B}}^\top$ we get the system

$$\mathbf{M}_L^{-1}\tilde{\mathbf{F}}\tilde{\boldsymbol{\lambda}} = \mathbf{M}_{\mathbf{L}}^{-1}\tilde{\mathbf{g}}, \tag{18}$$

with the notation

$$\tilde{\mathbf{F}} = \tilde{\mathbf{B}}(\tilde{\mathbf{K}}^{ef})^{-1}\tilde{\mathbf{B}}^\top, \quad \tilde{\mathbf{g}} = \tilde{\mathbf{B}}(\tilde{\mathbf{K}}^{ef})^{-1}\tilde{\mathbf{f}} - \tilde{\mathbf{c}} \ .$$

This system is solved by conjugate gradient method (PCG).

In every PCG iteration we need to compute $\tilde{\mathbf{x}} = (\tilde{\mathbf{K}}^{ef})^{-1}\tilde{\mathbf{b}}$, where $\tilde{\mathbf{b}} = \left[\begin{array}{cc} \mathbf{b}_0^\top & \mathbf{d}_0^\top \end{array} \right]^\top$ is a given vector. To do this we solve the system

$$\tilde{\mathbf{K}}^{ef}\tilde{\mathbf{x}} = \tilde{\mathbf{b}} \quad \Longleftrightarrow \quad \begin{bmatrix} \mathbf{K}^{ef} & \mathbf{B}_0^\top \\ \mathbf{B}_0 & \mathbf{O} \end{bmatrix} \begin{bmatrix} \mathbf{x}_0 \\ \boldsymbol{\mu}_0 \end{bmatrix} = \begin{bmatrix} \mathbf{b}_0 \\ \mathbf{d}_0 \end{bmatrix} \tag{19}$$

by a TFETI based solver again. We use a notation $\tilde{\mathbf{x}} = \left[\begin{array}{cc} \mathbf{x}_0^\top & \boldsymbol{\mu}_0^\top \end{array} \right]^\top$. Using

$$\mathbf{x}_0 = (\mathbf{K}^{ef})^{-1}(\mathbf{b}_0 - \mathbf{B}_0^\top \boldsymbol{\mu}_0), \tag{20}$$

and substituting it into the second equation in (19) we get

$$\mathbf{F}_0\boldsymbol{\mu}_0 = \mathbf{g}_0, \tag{21}$$

where

$$\mathbf{F}_0 = \mathbf{B}_0(\mathbf{K}^{ef})^{-1}\mathbf{B}_0^\top, \qquad \mathbf{g}_0 = \mathbf{B}_0(\mathbf{K}^{ef})^{-1}\mathbf{b}_0 - \mathbf{d}_0.$$

Equation (21) is solved by a direct solver.

5 NUMERICAL EXPERIMENTS

Our benchmark is a transient problem of 3D elastic box Ω of size $a = 10$ [mm] depicted in Figure 3. Material constants are defined by the Young modulus $= E = 2.1 \cdot 10^5$ [MPa], the Poisson ratio $\nu = 0.3$, and the density $\rho = 7.85 \cdot 10^{-9}$ [ton/mm^3]. The displacements (the result of the static problem, the box is loaded by pressure load) are prescribed as an initial condition of the transient problem.

We decompose the box into s subdomains, for each we have regular rectangular discretization with 11^3 nodes. For the time discretization, we use Algorithm 1 with the constant time step $\Delta = 10^{-4}$ and solve the problem of box oscillations. The iteration counts in the first time step are depicted in Tables 1 and 2.

Remark: To compute larger problems in effective way we plan to use a Hybrid TFETI implementation [2], where the inner dual problem is in a block diagonal form and so can be easily parallelized.

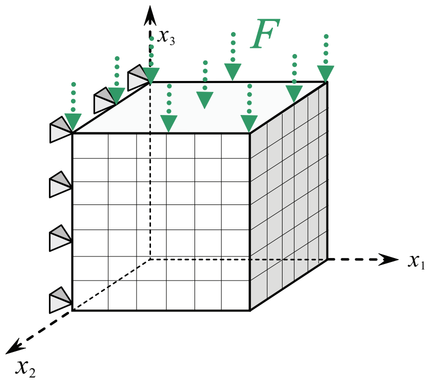

Figure 3: Transient problem of 3D elastic box.

Table 1: Iteration counts for CG and PCG with lumped preconditioner. The positions of fixing nodes (corners) are in Gauss points, 4 nodes per face.

s	CG	PCG(lumped)
2^3	55	38
4^3	72	59
6^3	84	64

Table 2: Iteration counts for CG and PCG with lumped preconditioner. The positions of fixing nodes (corners) are generated via METIS, 5 nodes per face.

s	CG	PCG(lumped)
2^3	44	35
4^3	62	42
6^3	67	49

6 ACKNOWLEDGMENTS

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REFERENCES

[1] Z. Dostal, T. Kozubek, T. Brzobohaty, A. Markopoulos, O. Vlach, *Scalable TFETI with optional preconditioning by conjugate projector for transient frictionless contact problems of elasticity*, Computer Methods in Applied Mechanics and Engineering, Volumes 247248, 1 November 2012, Pages 37-50

[2] M. Jarosova, T. Kozubek, M. Mensik, A. Markopoulos, *Hybrid Total FETI method.* ECCOMAS 2012 - European Congress on Computational Methods in Applied Sciences and Engineering, e-Book Full Papers, Vienna University of Technology, 2012, s. 6653-6663. 978-3-9503537-0-9

[3] Z. Dostal, D. Horak, R. Kucera, *Total FETI - an easier implementable variant of the FETI method for numerical solution of elliptic PDE.* Communications in Numerical Methods in Engineering, 22(2006), 12, pp. 1155-1162.

[4] Z. Dostál, T. Kozubek, V. Vondrák, T. Brzobohatý and A. Markopoulos, *Scalable TFETI algorithm for the solution of multibody contact problems of elasticity*,

[5] C. Farhat, F-X. Roux, *An unconventional domain decomposition method for an efficient parallel solution of large-scale finite element systems*, SIAM J. Sci. Stat. Comput. 13: 379396, 1992

[6] C. Farhat, M. Lesoinne, P. LeTallec, K. Pierson, and D. Rixen, *FETI-DP: A dual-primal unified FETI method - part I: A faster alternative to the two-level FETI method.* International Journal for Numerical Methods in Engineering, 50:1523-1544, 2001

[7] J. Mandel and R. Tezaur, *On the convergence of a dual-primal substructuring method.* Numer. Math., 88:543-558, 2001

[8] K. J. Bathe, *Finite Element Procedures*, Prentice Hall, 1996.