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ON SOME ASPECTS OF A POSTERIORI ERROR ESTIMATION IN THE MULTIPOINT MESHLESS FDM

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Abstract. The higher order multipoint meshless finite difference method (MFDM) is considered in this paper. The new approach is based on arbitrary irregular meshes, the moving weighted least squares approximation and the local or various global formulations of boundary value problems. A priori and a posteriori errors constitute the important part of engineering problem analysis. The paper is focused on application of the multipoint method to a posteriori estimation of the solution and residual errors. The multipoint approach provides high precision results that may be used as a reference solution in global or local error estimators. A variety of 1D and 2D tests done confirm high quality of a posteriori error estimation based on the multipoint MFDM.

1 INTRODUCTION

Reliable error estimation of a boundary value problems solution constitutes an important part of numerical analysis. Both a priori and a posteriori errors [1] may be considered here, though the last ones may be evaluated only after a solution of the problem is found. As the exact solution is unknown, the true error of the obtained rough solution is evaluated by using an improved solution as the reference one. The reference solution needed here may be calculated by using an adaptive (*h*-type) solution approach, or may be achieved without raising the number of nodes in the mesh – by rising the approximation order (*p*-type). Among various methods providing the higher order (HO) approximations, the new multipoint meshless finite difference method (MFDM) may be used. Therefore, the formulation and implementation an a posteriori error estimation approach based on the multipoint MFDM is the objective of this research.

The main idea of multipoint concept was introduced a long time ago by Collatz [2] as improvement of FDM based on the higher order approximation. The order of approximation of a searched function is raised by assuming additional degrees of freedom at the stencil nodes, including e.g. the right-hand side value of the considered differential equation. In this way, the multipoint FD formula takes into account a combination of unknown function values together with a combination of additional degrees of freedom, e.g. right hand side of PDE. The higher order FD operator is generated using the same set of nodes as in the non multipoint case. This fact is advantage in comparison with other HO methods (e.g. so called defect (deferred) correction approach [3], based on increasing the number of nodes included into stencil) due to less calculations needed. Such approach improves the quality of solutions of boundary value problems analyzed by means of the MFDM.

The original multipoint Collatz concept (regular meshes, interpolation, local formulation of b.v. problems) has been reformulated by the authors [4-6] and extended to the fully automatic multipoint meshless FDM. Arbitrarily distributed clouds of nodes, as well as application of the moving weighted least squares (MWLS) approximation [7], make possible in this way to develop the multipoint MFDM solution approach. Besides development of the multipoint meshless FDM for the analysis of b.v. problems given in the local (strong) formulation, the multipoint method was also extended to the global and global-local formulations including the minimum of the total potential energy, the variational Galerkin and the meshless local Petrov-Galerkin (MLPG) [8].

Due to its high quality results, the multipoint method may be also used to provide reference solutions needed for the global or local error estimation. A posteriori error analysis of the multipoint MFDM results may be applied for two purposes: to examine the solution quality and to generate series of adaptive meshes. In this paper attention is laid upon formulation and preliminary application of the multipoint MFDM to a posteriori solution error estimation.

2 MULTIPOINT MFDM SOLUTION APPROACH

Let us consider the local (strong) formulation of boundary value problems given in a domain Ω for the n-th order ODE (PDE) with appropriate b.c.

$$\begin{cases} \mathcal{L}u = f, \\ \mathcal{G}u = g, \end{cases} \qquad u = u(P), \quad \begin{matrix} \text{for} & P \in \Omega \\ \text{for} & P \in \partial\Omega \end{matrix},$$

where $\mathcal{L}, \mathcal{G}$ are differential operators or an equivalent global (weak) one formulated as a variational principle

$$b(u, v) = l(v), \qquad \forall\ v \in V \ .$$

Here b is a bilinear functional dependent on the test function v and solution u of the considered b.v. problem, V is the space of test functions, l is a linear form dependent on v.

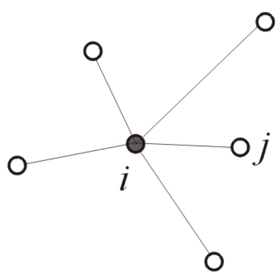

Figure 1: MFD star (stencil)

Using the FDM or MFDM discretization based on the selected FD stars (Fig. 1) with respect to a central node *i*, the classical difference operator *Lu* is presented in the following form

$$\mathcal{L}u_i \approx Lu_i \equiv \sum_{j(i)} c_j u_j = f_i \quad \Rightarrow \quad Lu_i = f_i \,. \tag{1}$$

In the multipoint formulation, the MFDM difference operator *Lu* is obtained by using additional degrees of freedom at nodes. Instead of the function value at the central node only, a combination of the right-hand side values f_i of the considered differential equation at each stencil node is applied:

$$\mathcal{L}u_i \approx Lu_i \equiv \sum_{j(i)} c_j u_j = \sum_{j(i)} \alpha_j f_j \quad \Rightarrow \quad Lu_i = Mf_i \,. \tag{2}$$

Moreover arbitrarily distributed clouds of nodes may be used then. This is the basic formula for the so called multipoint specific formulation. The following notation has been used here: *j* – is a number of a node in the selected FD star, Mf_i – a combination of the equations right-hand side nodal values, f_i – may present value of the whole operator $\mathcal{L}$ or its part only, e.g. a specific derivative. In general *L* may be either referred to the left side of differential eqs or to the integrand in the global formulation of b.v. problem, and to the boundary conditions.

In the multipoint formulation, the difference operators *L* and *M* are obtained by means of the Taylor series expansion of unknown function *u* and additional degrees of freedom, e.g. right hand side part *f*, including derivatives of the higher order.

3 ERROR ANALYSIS

Examination of the solution of b.v. problems quality and mesh adaptation is based on an a posteriori error analysis [1]. Because, in general, the exact solution is unknown, the higher precision result – in this case the HO multipoint MFDM solution – can be used as the reference solution to evaluate the error. Several types of error estimations are currently available [9]. Three of them applied to multipoint approach are briefly discussed below.

3.1 Hierarchic estimators

The hierarchic solution estimators are based on the comparison of calculated results with reference solution obtained using *h*-, *p*-, or *hp*- approach. The multipoint method may be successfully applied to this purpose.

When the exact analytical solution u^T of a boundary value problem is known (e.g. in benchmark problems) and rough numerical solution u^L using standard MFDM approach (1) is found, the true solution error could be calculated as follows

$$e^{TL} = u^T - u^L \,. \tag{3}$$

When the multipoint approach (2) is applied, the improved higher order solution u^H is obtained and used as the reference solution. One may then estimate the true solution error (3) as follows

$$e^{HL} = u^H - u^L \approx e^{TL} \,. \tag{4}$$

Moreover, the exact higher order solution error e^{TH}

$$e^{TH} = u^T - u^H . \tag{5}$$

may be also estimated by using the multipoint method with different orders of approximation e.g., $p1$ and $p2 > p1$. In this case

$$e^{HH} = u^{H(p1)} - u^{H(p2)} \approx e^{TH} .$$

The error estimation quality may be measured by means of the so called effectivity index.

$$I_{\text{eff}} = 1 + \left| \left\| e^{HL} \right\| - \left\| e^{TL} \right\| \right| / \left\| e^{TL} \right\| . \tag{6}$$

The index is based on comparison of the true low order e.g. e^{TL} and estimated e^{HL} errors, and $I_{eff} = 1$ for the true solution.

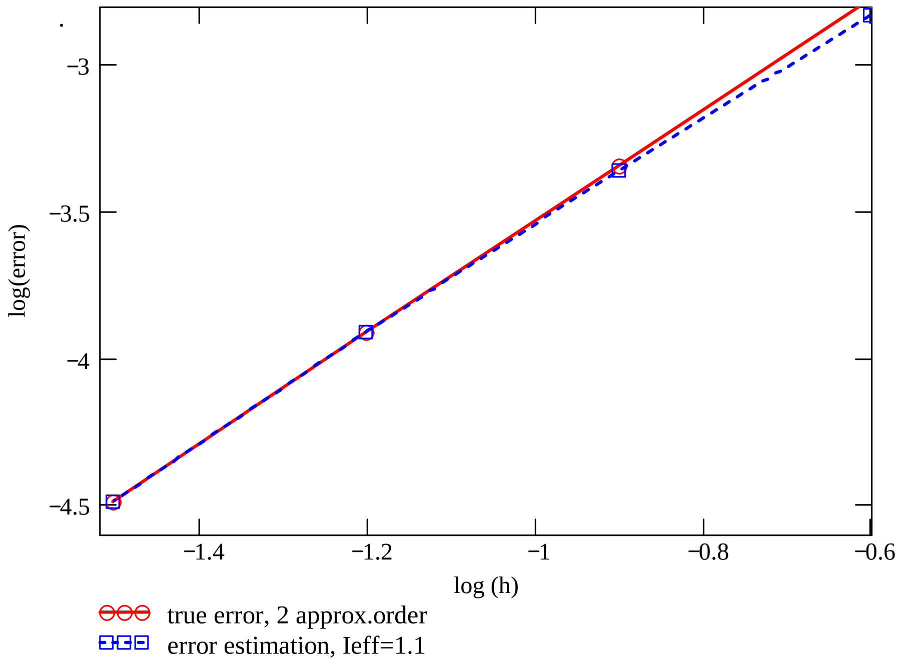

Figure 2: Comparison of the error estimation e^{HL} and exact error e^{TL} for low order solution. Test1. Poisson's b.v.p. with Dirichlet b.c. $\nabla^2 u = f(x, y)$, $0 \le x, y \le 1$, $f(x, y) = -2 \cdot \sin(x + y)$

Several tests done for the multipoint approache confirm the observation that the calculated values of the above effectivity index were close to 1 (Fig. 2).

The global error may be measured using the energy norm, computed either as the continuous $L^2(\Omega)$ norm

$$\left\| e \right\|_{E(\Omega)} = \left(\frac{1}{\Omega} \int_\Omega b(u - \tilde{u}, u - \tilde{u}) d\Omega \right)^{1/2}$$

or the discrete l^2 one

$$\|e\|_2 = \left(\frac{1}{N} \sum_{i=1}^{N} (u_i - \tilde{u}_i)^2 \right)^{1/2} .$$

Here u is the true solution and $\tilde{u}$ is an approximate rough one.

3.2 Residual estimators

The residual estimators use either explicit residual errors of high order r^H, or low order r^L as follows

$$r^H = Lu^H - f \, , \tag{7}$$

$$r^L = Lu^L - f \, ,$$

or equivalent implicit ones (not specified here). Each of them provides a quality measure of the higher (multipoint approach) or lower (standard MFDM) order solution error.

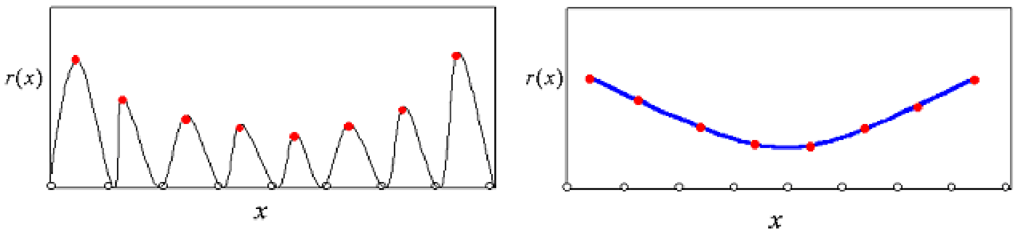

Figure 3: Residual error plot

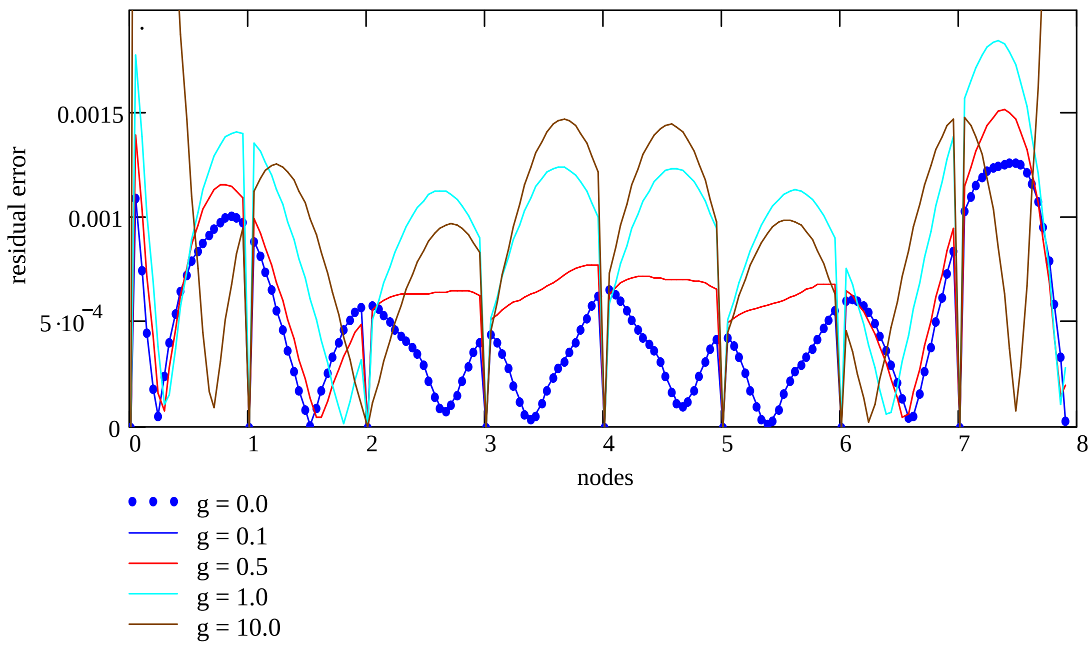

Figure 4: Residual error distribution using 20 points between nodes. The influence of the MWLS weight factor *g*

Using approximated higher order solution u^H, defined at the nodes, one may calculate the residuum between the nodes (at the nodes the residual error is equal to zero due to the collocation requirement imposed). At the middle of this distance, the residuum is expected to reach its maximum value (Fig. 3). However, the tests done showed that the error distribution essentially depends on the smoothing parameter g used in the MWLS (Fig. 4).

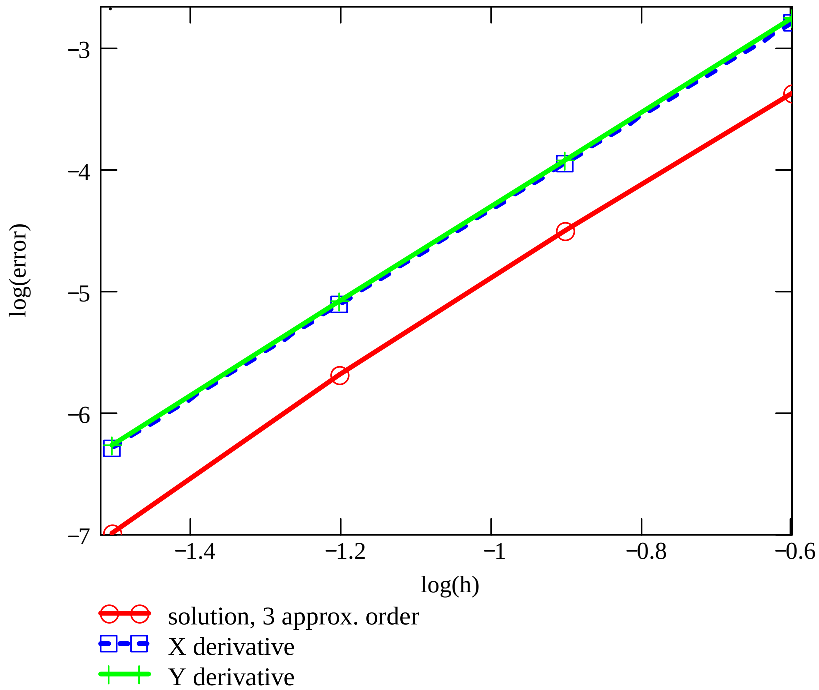

Figure 5: Convergence rates of solution and its derivatives exact errors. Test1.

3.3 Smoothing estimators

Smoothing estimators (well known as Zienkiewicz-Zhu one [10]) is based on the comparison between the recovered and the reference derivatives (e.g. stresses). Using multipoint method the unknown function derivatives up to order p are also obtained without any additional computational cost besides solution. Then the higher order approximation of derivatives may be used instead of the recovered ones. It is worth noticing here, that in multipoint approach the convergence rate for solution is of the same order as for its derivatives (Fig. 5).

4 FINAL REMARKS

The higher order multipoint meshless finite difference method based on arbitrary irregular clouds of nodes, moving weighted least squares approximation and the global, local or global-local formulations of boundary value problems, was considered. The paper focused on a posteriori estimation of the global or local solution and residual errors based on the multipoint MFDM.

Due to high quality of its results, the multipoint method may be also used to develop the reference solutions needed for a posteriori error (global and local) estimation. A posteriori error analysis of the multipoint MFDM may be applied for two purposes: examination of the solution quality and generation of a series of adaptive meshes. Multiple preliminary tests confirm high quality of a posteriori error estimation based on the multipoint MFDM.

Further development of the use of the multipoint meshless FDM a priori and a posteriori error analysis is planned.

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