11th World Congress on Computational Mechanics (WCCM XI)
5th European Conference on Computational Mechanics (ECCM V)
6th European Conference on Computational Fluid Dynamics (ECFD VI)
E. Oñate, J. Oliver and A. Huerta (Eds)

THE INITIAL–BOUNDARY RIEMANN PROBLEM FOR THE SOLUTION OF THE COMPRESSIBLE GAS FLOW

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Key words: Compressible Gas Flow, the Riemann problem, Boundary Conditions

Abstract. The paper deals with a special way of the construction of boundary conditions for the compressible gas flow. The solution of the Riemann problem is used at the boundary. It can be shown, that the unknown one-side initial condition for this problem can be partially replaced by the suitable complementary condition. Authors work with such initial-boundary problem and suggest the complementary conditions by the preference of total quantities, in order to match the physically relevant data. Algorithms were coded and used within the own developed code for the solution of the Euler, NS, and the RANS equations, using the finite volume method. Numerical example shows good behavior of this boundary condition.

1 INTRODUCTION

Undoubtedly, the boundary conditions play important role in the computational fluid dynamics (CFD). We work with the system of equations describing non-stationary compressible turbulent fluid flow, i.e. the Reynolds-Averaged Navier-Stokes (RANS) equations in 2D and 3D. We focus on the numerical solution of these equations, and on the boundary conditions. We suggest the original approach to the boundary conditions. The aim is to satisfy the conservation laws in the close vicinity of the boundary. Usually, the boundary problem is being linearized, or roughly aproximated. The inaccuracies implied by these simplifications may be small, but it has a huge impact on the solution in the whole studied area, especially for the non-stationary flow. In our approach we try to be as exact as possible. Therefore we use the analysis of the so-called Riemann problem for the 2D/3D split Euler equations in order to construct the boundary values (for the density, velocity, pressure). We solve the boundary modifications of this initial-value problem. We show the analysis leading to our results using the well-known finite volume method (FVM) to discretize the analytical problem, represented by the system of the equations in generalized (integral) form. To apply this method we split the area of the interest into the elements, and we construct a piecewise constant solution in time. The crucial problem of this method lies in the evaluation of the so-called fluxes through the edges/faces Γ_{ij} of the particular elements. The state values in the vicinity of the edge Γ_{ij} are known at time instant t_k, and these values form the initial conditions (LIC - left-hand side, and RIC - right-hand side) for the so-called Riemann problem for the 2D/3D split Euler equations. The exact (entropy weak) solution of this problem cannot be expressed in a closed form, and has to be computed by an iterative process (to given accuracy). Therefore various approximations of this solution are usually analyzed. At the boundary faces we deal with the local modified Riemann problem, where the LIC is given, while the RIC is not known. In some cases (far field boundary) it is wise to choose the RIC here as the solution of the local Riemann problem with given far field values, which gives better results than the solution of the linearized Riemann problem, see [1]. Another boundary condition based on the exact Riemann problem solution, simulating the impermeable wall on move, was shown in [3, pages 221-225], where the RIC is constructed in a special way, in order to obtain the desired solution. Using the analysis of the Riemann problem we show, that the RIC for the local problem can be partially replaced by the suitable complementary condition. Some of the suggested boundary conditions were shown in [2] (by preference of pressure, temperature, velocity,...). Here we focus on the inlet boundary condition conserving the total quantities and the direction of the velocity. We complement the boundary problem suitably, and we show the analysis of the resulting uniquely-solvable modified Riemann problem. We construct own algorithm for the solution of this boundary problem. It can be used within various methods in CFD. The algorithm was coded and used within our own developed code for the solution of the compressible gas flow (the Euler, NS, and RANS equations).

2 THE RIEMANN PROBLEM FOR THE SPLIT EULER EQUATIONS

In order to approximate the state values at the particular edges/faces of the mesh (at each time instant), we use the solution of the so-called Riemann problem for the split Euler equations. Using the rotational invariance of the equations describing the fluid flow, the system is expressed in a new Cartesian coordinate system $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ with the origin at the center of the gravity of the edge of interest Γ_{ij} and with the new axis $\tilde{x}_1$ in the direction of the normal of the edge. Then the derivatives with respect to $\tilde{x}_2, \tilde{x}_3$ are neglected, and we get the so-called split 3D Euler equations, see [5, page 138]:

$$\frac{\partial \boldsymbol{q}}{\partial t} + \frac{\partial \boldsymbol{f}_1(\boldsymbol{q})}{\partial \tilde{x}_1} = 0. \tag{1}$$

Here t denotes the time. $\boldsymbol{q} = \boldsymbol{q}(\tilde{x}, t) = (\varrho,\ \varrho u,\ \varrho v,\ \varrho w,\ E)^T$ is the state vector, $\boldsymbol{f}_1 = (\varrho u,\ \varrho uu + p, \varrho uv, \varrho uw,\ (E + p)\, u)^T$ are the inviscid fluxes, $\boldsymbol{v} = (u, v, w)^T$ denotes the velocity vector in the local coordinate system $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$, ϱ is the density, p the pressure, θ the absolute temperature, $E = \varrho e + \frac{1}{2}\varrho \boldsymbol{v}^2$ the total energy. For the specific internal energy $e = c_v\theta$ we assume the caloric equation of state $e = p/\varrho(\gamma - 1)$, c_v is the specific heat at constant volume, $\gamma > 1$ is called the *Poisson adiabatic constant*. The system (1) is considered for $(\tilde{x}_1, t)$ in the set $Q_\infty = (-\infty, \infty) \times (0, +\infty)$.

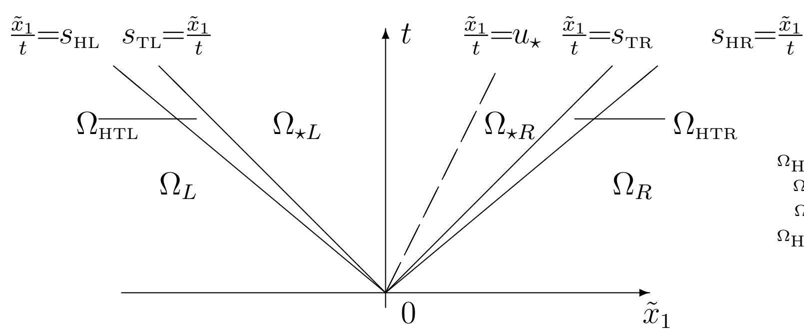
Let us suppose, that the state values (density, velocity, pressure) are known in the close vicinity of the edge Γ_{ij} at the time instant t_k. These two states form the initial condition for the problem (1).

$$\boldsymbol{q}(\tilde{x}_1, 0) = \boldsymbol{q}_L = \mathbb{Q}\, \boldsymbol{w}_i^k, \qquad \tilde{x}_1 < 0, \tag{2}$$

$$\boldsymbol{q}(\tilde{x}_1, 0) = \boldsymbol{q}_R = \mathbb{Q}\, \boldsymbol{w}_j^k, \qquad \tilde{x}_1 > 0. \tag{3}$$

This problem (1),(2),(3) is the so-called Riemann problem for the *split* Euler equations. The solution of this problem at time axis is the desired solution (density, velocity, pressure) at the edge Γ_{ij}, and can be later used within the finite volume method.

It is a characteristic feature of the hyperbolic equations, that there is a possible raise of discontinuities in solutions, even in the case when the initial conditions are smooth, see [4, page 390], therefore by solution we mean the so-called *entropy weak solution* to this problem. The analysis to the solution of this problem can be found in many books, i.e. [5], [4], [3]. The general theorem on the solvability of the Riemann problem can be found in [5, page 88]. Here we summarize, that the problem has a unique solution for certain choice of the initial conditions. This solution can be written for $t > 0$ in the similarity form $\boldsymbol{q}(\tilde{x}_1, t) = \tilde{\boldsymbol{q}}(\tilde{x}_1/t)$, where $\tilde{\boldsymbol{q}}(\tilde{x}_1/t) : \mathbb{R} \to \mathbb{R}^3$ ([5, page 82]). The solution is piecewise **smooth** and its general form can be seen in Fig. 1, where the system of half lines is drawn. These half lines define regions, where $\boldsymbol{q}$ is either constant or given by a smooth



$$\begin{aligned}
\Omega_L &= \{(\tilde{x}_1, t);\ \tfrac{\tilde{x}_1}{t} < s_{\mathrm{HL}};\ t > 0\},\\
\Omega_{\mathrm{HTL}} &= \{(\tilde{x}_1, t);\ s_{\mathrm{HL}} < \tfrac{\tilde{x}_1}{t} < s_{\mathrm{TL}};\ t > 0\},\\
\Omega_{\star L} &= \{(\tilde{x}_1, t);\ s_{\mathrm{TL}} < \tfrac{\tilde{x}_1}{t} < u_\star;\ t > 0\},\\
\Omega_{\star R} &= \{(\tilde{x}_1, t);\ u_\star < \tfrac{\tilde{x}_1}{t} < s_{\mathrm{TR}};\ t > 0\},\\
\Omega_{\mathrm{HTR}} &= \{(\tilde{x}_1, t);\ s_{\mathrm{TR}} < \tfrac{\tilde{x}_1}{t} < s_{\mathrm{HR}};\ t > 0\},\\
\Omega_R &= \{(\tilde{x}_1, t);\ s_{\mathrm{HR}} < \tfrac{\tilde{x}_1}{t};\ t > 0\}.
\end{aligned}$$

Figure 1: Structure of the solution of the Riemann problem (1),(2),(3)

function. Let us define the open sets called **wedges** Ω_L, Ω_{HTL}, $\Omega_{\star L}$, $\Omega_{\star R}$, Ω_{HTR}, Ω_R, see Fig. 1. We will refer to the set Ω_{HTL} as to the left wave, and the set Ω_{HTR} will be called the right wave. The solution in Ω_L, $\Omega_{\star L}$, $\Omega_{\star R}$, Ω_R is constant (see e.g. [5, page 128]), while in Ω_{HTL} and in Ω_{HTR} it is continuous. Let us denote $\boldsymbol{q}|_{\Omega_L} = \boldsymbol{q}_L$, $\boldsymbol{q}|_{\Omega_{\star L}} = \boldsymbol{q}_{\star L}$, $\boldsymbol{q}|_{\Omega_{\star R}} = \boldsymbol{q}_{\star R}$, $\boldsymbol{q}|_{\Omega_R} = \boldsymbol{q}_R$, $\boldsymbol{q}|_{\Omega_{HTL}} = \boldsymbol{q}_{HTL}$, $\boldsymbol{q}|_{\Omega_{HTR}} = \boldsymbol{q}_{HTR}$.

The exact solution of the Riemann problem has three waves in general, illustrated in Fig. 1. The wedges Ω_L and $\Omega_{\star L}$ are separated by the left wave (either 1-*shock wave*, or

1-*rarefaction wave*). There is a *contact discontinuity* between the regions $\Omega_{\star L}$ and $\Omega_{\star R}$. Wedges $\Omega_{\star R}$ and Ω_R are separated by the right wave (either 3-*shock wave*, or 3-*rarefaction wave*). The solution for the primitive variables can be described as follows:

$$(\varrho,u,v,w,p)|_{\Omega_L} = (\varrho_L,u_L,v_L,w_L,p_L), \qquad (\varrho,u,v,w,p)|_{\Omega_{\star R}} = (\varrho_{\star R},u_\star,v_R,w_R,p_\star),$$
$$(\varrho,u,v,w,p)|_{\Omega_{\star L}} = (\varrho_{\star L},u_\star,v_L,w_L,p_\star), \qquad (\varrho,u,v,w,p)|_{\Omega_R} = (\varrho_R,u_R,v_R,w_R,p_R).$$

The folowing relations for these variables hold:

$$u_\star = u_L + \begin{cases} -(p_\star - p_L)\left(\dfrac{\frac{2}{(\gamma+1)\varrho_L}}{p_\star + \frac{\gamma-1}{\gamma+1}p_L}\right)^{\frac{1}{2}}, & p_\star > p_L \\ \frac{2}{\gamma-1}a_L\left[1-\left(\frac{p_\star}{p_L}\right)^{(\gamma-1)/2\gamma}\right], & p_\star \le p_L \end{cases} \qquad (4)$$

$$u_\star = u_R + \begin{cases} (p_\star - p_R)\left(\dfrac{\frac{2}{(\gamma+1)\varrho_R}}{p_\star + \frac{\gamma-1}{\gamma+1}p_R}\right)^{\frac{1}{2}}, & p_\star > p_R, \\ -\frac{2}{\gamma-1}a_R\left[1-\left(\frac{p_\star}{p_R}\right)^{(\gamma-1)/2\gamma}\right], & p_\star \le p_R \end{cases} \qquad (7)$$

$$\varrho_{\star L} = \begin{cases} \varrho_L \dfrac{\frac{\gamma-1}{\gamma+1}\frac{p_L}{p_\star}+1}{\frac{p_L}{p_\star}+\frac{\gamma-1}{\gamma+1}}, & p_\star > p_L \\ \varrho_L\left(\frac{p_\star}{p_L}\right)^{\frac{1}{\gamma}}, & p_\star \le p_L \end{cases} \qquad (5)$$

$$\varrho_{\star R} = \begin{cases} \varrho_R \dfrac{\frac{p_\star}{p_R}+\frac{\gamma-1}{\gamma+1}}{\frac{\gamma-1}{\gamma+1}\frac{p_\star}{p_R}+1}, & p_\star > p_R \\ \varrho_R\left(\frac{p_\star}{p_R}\right)^{\frac{1}{\gamma}}, & p_\star \le p_R \end{cases} \qquad (8)$$

$$s^1_{TL} = \begin{cases} u_L - a_L\sqrt{\frac{\gamma+1}{2\gamma}\frac{p_\star}{p_L}+\frac{\gamma-1}{2\gamma}}, & p_\star > p_L \\ u_\star - a_L\left(\frac{p_\star}{p_L}\right)^{(\gamma-1)/2\gamma}, & p_\star \le p_L \end{cases} \qquad (6)$$

$$s^3_{TR} = \begin{cases} u_R + a_R\sqrt{\frac{\gamma+1}{2\gamma}\frac{p_\star}{p_R}+\frac{\gamma-1}{2\gamma}}, & p_\star > p_R \\ u_\star + a_R\left(\frac{p_\star}{p_R}\right)^{(\gamma-1)/2\gamma}, & p_\star \le p_R \end{cases} \qquad (9)$$

Here $a_L = \sqrt{\gamma p_L/\varrho_L}$, $a_R = \sqrt{\gamma p_R/\varrho_R}$, and γ denotes the adiabatic constant. Further s^1_{TL} denotes "unknown left wave speed", s^3_{TR} "unknown right wave speed". Note, that the system (4) - (9) is the system of 6 equations for 6 unknowns $p_\star$,$u_\star$,$\varrho_{\star L}$ $\varrho_{\star R}$,s^1_{TL},s^3_{TR}. The solution of this system leads to a nonlinear algebraic equation, and one cannot express the analytical solution of this problem in a closed form. The problem has a solution only if the *pressure positivity condition* is satisfied

$$u_R - u_L < \frac{2}{\gamma-1}(a_L + a_R). \qquad (10)$$

We will use some of these relations to construct and solve the initial-boundary value problem which will be the original result of our work.

Remarks

- Once the pressure $p_\star$ is known, the solution on the left-hand side of the contact discontinuity depends only on the left-hand side initial condition (2). And similarily, with $p_\star$ known, only the right-hand side initial condition (3) is used to compute the solution on the right-hand side of the contact discontinuity.

- The solution in $\Omega_L \cup \Omega_{HTL} \cup \Omega_{\star L}$ (across 1 wave)
  There are three unknowns in the region $\Omega_{\star L}$. It is the density $\varrho_{\star L}$, the pressure $p_\star$, and the velocity $u_\star$. Also the speed s^1_{TL} of the left wave determining the position of the region Ω_{HTL} is not known. The solution components in $\Omega_L \cup \Omega_{HTL} \cup \Omega_{\star L}$ region must satisfy the system of equations (4)-(6). It is a system of three equations for four unknowns. We have to add another equation in order to get the uniquely solvable system in $\Omega_L \cup \Omega_{HTL} \cup \Omega_{\star L}$.

- The equation (4) can be written as the *equation for pressure*, see [8]

$$p_\star = E_1(u_\star), \quad E_1(u) = \begin{cases} \frac{2p_L + \frac{\gamma+1}{2}\varrho_L(u_L-u)^2 + (u_L-u)^2\sqrt{4\varrho_L\gamma p_L + \varrho_L^2(\frac{\gamma+1}{2})^2(u_L-u)^2}}{2}, & u < u_L, \\ p_L\left(\frac{-u+u_L+\frac{2}{\gamma-1}a_L}{\frac{2}{\gamma-1}a_L}\right)^{\frac{2\gamma}{\gamma-1}}, & u_L \le u < u_L + \frac{2}{\gamma-1}a_L. \end{cases} \tag{11}$$

3 BOUNDARY CONDITION BY PREFERENCE OF TOTAL QUANTITIES AND DIRECTION OF VELOCITY

Here we will construct the boundary state $\varrho_\Gamma, \boldsymbol{v}_\Gamma, p_\Gamma$ (at the boundary face Γ), following the procedure and notation described above in section 2. Let the state vector in the vicinity of the face at the given time instant be known, we denote this known state as $\varrho_G, \boldsymbol{v}_G, p_G$. Our aim is to find the boundary state values with the known total pressure p_o, total temperature θ_o, and the direction of the velocity $\boldsymbol{d}$, if possible. The system of conservation laws must be satisfied for these values. We use the transformation of this problem into a local coordinate system $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ (shown in [8]), with the axis $\tilde{x}_1$ being in the opposite direction of the prescribed velocity direction, given by $\boldsymbol{n} = -\boldsymbol{d}$, $|\boldsymbol{d}| = 1$. The situation is depicted in figure 3.

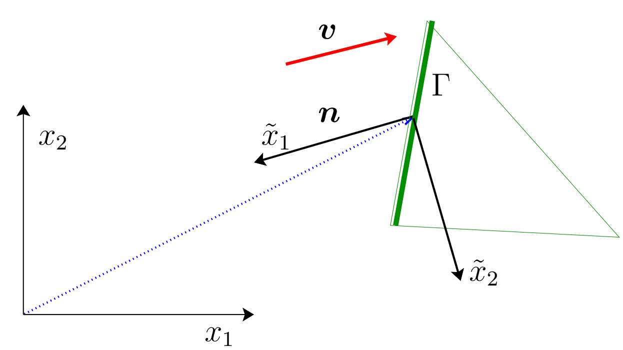


Figure 2: Coordinate transformation for the boundary edge Γ in 2D. The axis $\tilde{x}_1$ is in the opposite direction to the prescribed direction of the velocity.

Using such transformation, the density, 1st velocity component (in the direction of the $\boldsymbol{n}$), and the pressure in the vicinity of the face area are $\varrho_L = \varrho_G$, $u_L = \boldsymbol{v} \cdot \boldsymbol{n}$, $p_L = p_G$. We neglect the derivatives with respect to $\tilde{x}_2, \tilde{x}_3$, and we are interested in the solution of the equations (1) in the vicinity of the chosen face. Our aim is to evaluate the boundary state $\boldsymbol{q}_B = \boldsymbol{q}(0,t)$. We add the following **complementary conditions**

$$p_\star = p_o\left(1 - \frac{\gamma-1}{2a_0^2}u_\star^2\right)^{\gamma/(\gamma-1)}, \tag{12}$$

$$\theta_{\star R} = \theta_o \left(1 - \frac{\gamma - 1}{2a_o^2} u_\star^2\right), \tag{13}$$

$$u_\star < 0,\ p_\star = p_R,\ v_R = 0,\ w_R = 0. \tag{14}$$

Here $\theta_o > 0, p_o > 0$ are given constants and $a_o^2 = \gamma R \theta_o$. Further R denotes the characteristic gas constant, and γ is the adiabatic constant. The equations (12),(13) are considered for the velocity $u_\star \in (-\sqrt{\frac{2a_o^2}{(\gamma-1)}}, \sqrt{\frac{2a_o^2}{(\gamma-1)}})$. The equations (13),(12) come from the idea, that the *total pressure* p_o and the *total temperature* θ_o are known (prescribed) in the region $\Omega_{\star R}$. The **boundary state $\boldsymbol{q}_B$** is the solution of the system (1),(2),(12),(13) at the half line $S_B = \{(\tilde{x}_1, t);\ \tilde{x}_1 = s_B\, t\ ;\ t > 0\}$, $s_B = 0$. At first, we resolve the velocity $u_\star$ in the *Star Region* $\Omega_{\star R} \cup \Omega_{\star L}$, introduced in Section 2.

Solution for the velocity $u_\star$

The condition (12) has the form

$$p_\star = E_{p_o}(u_\star),\ \text{where } E_{p_o}(u) = p_o \left(1 - \frac{\gamma - 1}{2a_o^2} u^2\right)^{\gamma/(\gamma-1)}. \tag{15}$$

We will use the analysis in Section 2, and discuss the condition (15) together with the equation for the pressure (11) and the condition (14). We get the velocity $u_\star$ solving the equation

$$F(u_\star) = 0, \tag{16}$$

where

$$F(u) = E_{p_o}(u) - E_1(u),\ \text{and } -\sqrt{\frac{2a_o^2}{(\gamma-1)}} < u < 0.$$

Here a_L denotes the speed of sound in Ω_L, the function $E_1(u)$ is defined in (11). There is no admissible solution of the system (1),(2), (12), (13) if $u_L + \frac{2}{\gamma-1} a_L < -\sqrt{\frac{2a_o^2}{(\gamma-1)}}$, the extreme value yields the zero pressure $p_\star = 0$. In practical applications, we may prescribe the closest velocity $u_\star = -\sqrt{\frac{2a_o^2}{(\gamma-1)}} + \epsilon$, with $\epsilon > 0$ being a small positive constant, though it is not the solution of the system. Let us analyze the function $F(u)$ for the 1-shock and for the 1-rarefaction wave separately.

1-shock wave

For the 1-shock wave it is $u < u_L$. We seek the root of the function

$$F(u) = p_o \left(1 - \frac{\gamma - 1}{2a_o^2} u^2\right)^{\gamma/(\gamma-1)} - p_L - \frac{\gamma + 1}{4} \varrho_L (u_L - u)^2 - \frac{(u_L - u)}{2} \sqrt{4 \varrho_L \gamma p_L + \varrho_L^2 (\frac{\gamma + 1}{2})^2 (u_L - u)^2}, \tag{17}$$

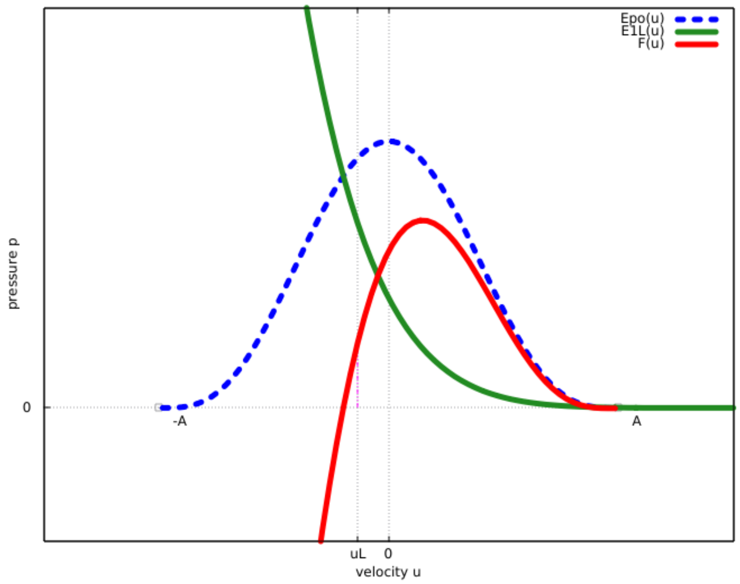


Figure 3: Example of the functions $E_{p_o}(u), E_{1L}(u), F(u)$ for the initial data $p_o = 101325, \theta_o = 273.15, p_L = 70000, \varrho_L = 1.25, u_L = -100$. Here $A = \sqrt{\frac{2a_o^2}{(\gamma-1)}}$.

We cannot use the form (17) in the case of $u_L \leq -\sqrt{\frac{2a_o^2}{(\gamma-1)}}$. The formulation (17) makes sense also for $u \in \{u_L, 0\}$. If $-\sqrt{\frac{2a_o^2}{(\gamma-1)}} < u_L$ then we seek the root of (17) in the interval $(-\sqrt{\frac{2a_o^2}{(\gamma-1)}}, \min\{u_L, 0\})$. It holds $F'(u) > 0$ in this interval, and $F(-\sqrt{\frac{2a_o^2}{(\gamma-1)}}) < 0$. If $F(\min\{u_L, 0\}) > 0$, then the solution $u_\star$ is unique.

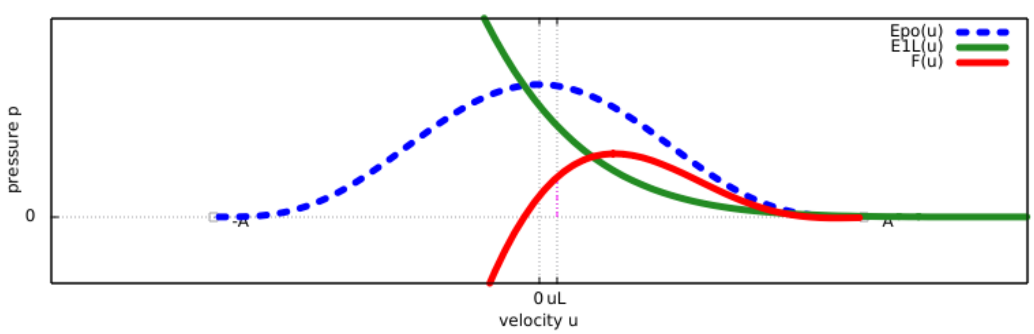


Figure 4: Example of the functions $E_{p_o}(u), E_{1L}(u), F(u)$ for the initial data $p_o = 101325, \theta_o = 273.15, p_L = 70000, \varrho_L = 1.25, u_L = 40$. In this example the problem (16) has one solution with the 1-shock wave. It is $u_\star < 0$.

1-rarefaction wave

For $u_L \le u < u_L + \frac{2}{\gamma-1}a_L$ we seek the root of the function

$$F(u) = p_o\left(1 - \frac{\gamma-1}{2a_o^2}u^2\right)^{\gamma/(\gamma-1)} - p_L\left(\frac{u_L - u + \frac{2}{\gamma-1}a_L}{\frac{2}{\gamma-1}a_L}\right)^{\frac{2\gamma}{\gamma-1}}, \tag{18}$$

in the interval of interest

$$u_L \le u < u_L + \frac{2}{\gamma-1}a_L, \quad -\sqrt{\frac{2a_o^2}{(\gamma-1)}} < u < 0.$$

For $u_L > 0$ there is no solution with the 1-rarefaction wave. If $F(\max\{u_L, -\sqrt{\frac{2a_o^2}{(\gamma-1)}}\}) < 0$ and $F(\min\{0, u_L + \frac{2}{\gamma-1}a_L\}) > 0$, then there is a unique root of the function (18), see [8]. It is

$$u_\star = \frac{2\left(u_L + \frac{2}{\gamma-1}a_L\right) - \sqrt{DIS}}{2\left[1 + \frac{2a_L^2}{(\gamma-1)a_o^2}\left(\frac{p_o}{p_L}\right)^{(\gamma-1)/\gamma}\right]}, \tag{19}$$

$$DIS = 4\left(\frac{p_o}{p_L}\right)^{(\gamma-1)/\gamma}\frac{2a_L^2}{(\gamma-1)a_o^2}\left[\frac{2a_o^2}{(\gamma-1)} + \frac{4a_L^2}{(\gamma-1)^2}\left(\frac{p_o}{p_L}\right)^{(\gamma-1)/\gamma} - \left(u_L + \frac{2a_L}{\gamma-1}\right)^2\right].$$

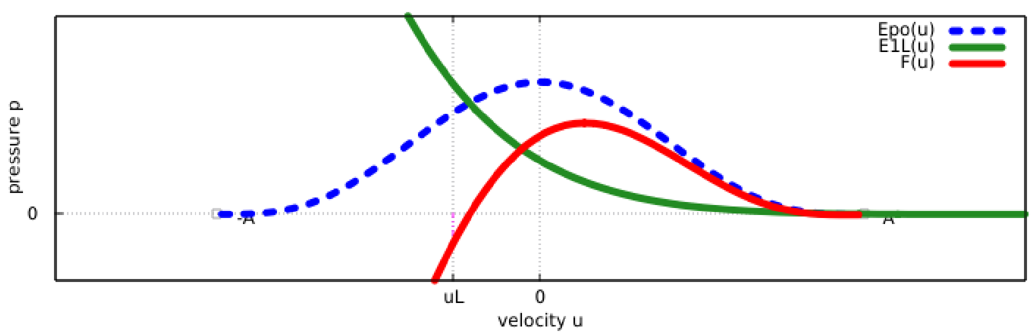


Figure 5: Example of the functions $E_{p_o}(u), E_{1L}(u), F(u)$ for the initial data $p_o = 101325, \theta_o = 273.15, p_L = 100000, \varrho_L = 1.25, u_L = -200$. In this example the problem (16) has one solution with the 1-rarefaction wave.

Let us summerize the above possibilities of the 1-shock and the 1-rarefaction wave. The problem (16) has a unique solution for the initial data satisfying

$$u_L + \frac{2}{\gamma-1}a_L > -\sqrt{\frac{2a_o^2}{(\gamma-1)}}, \quad F(\min\{0, u_L + \frac{2}{\gamma-1}a_L\}) > 0. \tag{20}$$

In the case of $u_L + \frac{2}{\gamma-1}a_L \leq -\sqrt{\frac{2a_o^2}{(\gamma-1)}}$ there is no solution of the system (1),(2),(12),(13) with (14). In this case we prescribe the velocity $u_\star = -\sqrt{\frac{2a_o^2}{(\gamma-1)}} + \epsilon$, with $\epsilon > 0$ being a small positive constant. If $F(\min\{0, u_L + \frac{2}{\gamma-1}a_L\}) \leq 0$ then the problem (16) does not have a negative solution, and for the practical applications we choose the velocity $u_\star = \min\{0, u_L + \frac{2}{\gamma-1}a_L\}$.

Solution in $(-\infty, \infty) \times (0, \infty)$

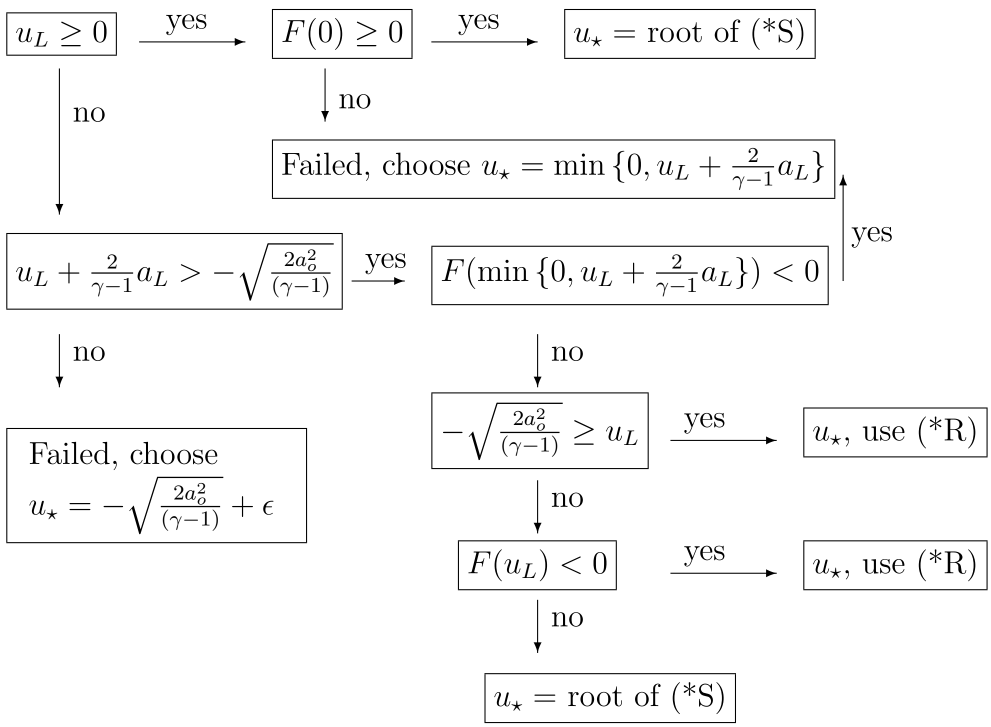
Knowing the velocity $u_\star$ we compute the pressure $p_\star$ using (15), and the density $\varrho_{\star L}$ using (6) uniquely. The density $\varrho_{\star R}$ can be computed using the equation of state and the equation for the *total temperature* (13) and the equation (15). It is

$$\varrho_{\star R} = \frac{p_\star}{R\theta_{\star R}} = \varrho_o \left(1 - \frac{\gamma - 1}{2a_o^2} u_\star^2\right)^{1/(\gamma-1)}. \tag{21}$$

Here $\varrho_o = \frac{p_o}{R\theta_o}$ denotes the *total density* in $\Omega_{\star R}$. Using (14), it is $u_\star = u_R$, and $\varrho_{\star R} = \varrho_R$, see [8]. $u_\star = u_R$, and $\varrho_{\star R} = \varrho_R$. This way the prescribed *total pressure* and *total temperature* are preferred also in Ω_R region.

The complementary conditions (12),(13),(14) yield the well-posed **inlet** boundary condition for the system (1),(2) only if (20) is satisfied. If (20) is not satisfied, then the *total quantities* cannot be preferred at the boundary. In the practical application we prescribe the closest admissible value for the velocity $u_\star$ and compute the boundary state accordingly, or we use the boundary condition preferring the static pressure $p_\star = p_o$, see [7, 8], in this case. The algorithm for the construction of the primitive variables $\varrho_B, u_B, v_B, w_B, p_B$ (in the local coordinate system) at the half-line S_B is shown in figure 6. The sought velocity in global coordinate system is $\boldsymbol{v}_\Gamma = (n_1 u_B, n_2 u_B, n_3 u_B)$, the density and the pressure is $\varrho_\Gamma = \varrho_B, \;\; p_\Gamma = p_B$.

1. compute $u_\star$



(*S) $F(u) = p_o\left(1 - \frac{\gamma-1}{2a_o^2}u^2\right)^{\gamma/(\gamma-1)} - p_L - \frac{\gamma+1}{4}\varrho_L(u_L-u)^2 - \frac{(u_L-u)}{2}\sqrt{4\varrho_L\gamma p_L + \varrho_L^2(\frac{\gamma+1}{2})^2(u_L-u)^2}$,

(*R) $u_\star = \dfrac{2\left(u_L + \frac{2}{\gamma-1}a_L\right) - \sqrt{DIS}}{2\left[1 + \frac{2a_L^2}{(\gamma-1)a_o^2}\left(\frac{p_o}{p_L}\right)^{(\gamma-1)/\gamma}\right]}$, $DIS = 4\left(\frac{p_o}{p_L}\right)^{(\gamma-1)/\gamma}\frac{2a_L^2}{(\gamma-1)a_o^2}\left[\frac{2a_o^2}{(\gamma-1)} + \frac{4a_L^2}{(\gamma-1)^2}\left(\frac{p_o}{p_L}\right)^{(\gamma-1)/\gamma} - \left(u_L + \frac{2a_L}{\gamma-1}\right)^2\right]$.

2. compute u_B, p_B, ϱ_B

$$u_B = u_\star,\ p_B = p_o\left(1 - \frac{\gamma-1}{2a_0^2}u_\star^2\right)^{\gamma/(\gamma-1)}, \varrho_B = \frac{p_o}{R\theta_o}\left(1 - \frac{\gamma-1}{2a_o^2}u_\star^2\right)^{1/(\gamma-1)}.$$

Figure 6: Algorithm for the solution of the problem (4)-(9),(12),(13),(14) at the half line $S_B = \{(0,t);\ t > 0\}$. In the case of failure, the problem doesn't have a solution, and the value for $u_\star$ is chosen.

4 NUMERICAL EXAMPLE

The following numerical example shows superior behavior of the suggested boundary condition. This is the test case of the the inviscid flow through the Double Circular Arc (DCA) blade cascade DCA08. The blades of this cascade are composed of two circular arcs with the relative thickness 8%. At the inlet we use the boundary condition conserving the total pressure $p_o = 101325$, the total temperature $\theta_o = 273.15$, and the direction of the velocity $\alpha_{IN} = 5.2$. At the outlet, the outlet boundary condition with the averaging technique described in [8], preferring the pressure $p_\star = 45722.351$ in average. The part of the inlet flow is supersonic. It can be seen in figure 7. (left), that the used inlet boundary condition does not reflect the shock waves into the computational area. The right picture shows that this boundary condition can be used on the shortened computational domain with the similar result. Constructed boundary condition is robust and accelerates the convergence of the method, it can be also used on the shortened domains. This is the original result of our work.

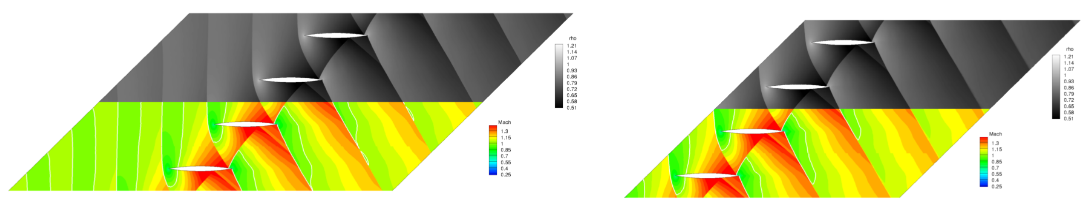


Figure 7: The compressible gas flow, transonic regime. Two computations with the same boundary data. The right picture shows the computation with the shortened inlet part. Inlet located left. Pressure isolines, density and Mach number isolines, highlighted Mach number 1.

5 CONCLUSIONS

This paper shows the analysis of the inlet boundary condition, using the original approach based on the analysis of the Riemann problem for the split Euler equations. In order to discretize the values at the boundary, the original initial-value problem was modified, as shown in [6]. The unknown right-hand side initial-value condition was replaced by the suitable conditions. The algorithms for the solution of the boundary problems were coded and implemented into the own-developed software for the solution of the compressible (laminar or turbulent) gas flow (the Euler equations, the Navier-Stokes equations, the Reynolds-Averaged Navier-Stokes equations) in 2D and 3D. Numerical example of the transonic cascade flow was shown.

Acknowledgment

This result originated with the support of Ministry of Industry and Trade of Czech Republic for the long-term strategic development of the research organisation. The authors acknowledge this support.

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