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CROSS-SECTIONAL ANALYSIS OF PRE-TWISTED THICK BEAMS USING VARIATIONAL ASYMPTOTIC METHOD

MAQSOOD M. AMEEN*, **DINESHKUMAR HARURSAMPATH**†

Non-linear Multifunctional Composites - Analysis and Design (NMCAD) Lab
Department of Aerospace Engineering, Indian Institute of Science
Bangalore-560012, India
*e-mail: maqsood@aero.iisc.ernet.in

†e-mail: dinesh@aero.iisc.ernet.in - Web page: www.aero.iisc.ernet.in/users/dinesh

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Abstract. An asymptotically-exact methodology is presented for obtaining the cross-sectional stiffness matrix of a pre-twisted moderately-thick beam having rectangular cross sections and made of transversely isotropic materials. The anisotropic beam is modeled from 3-D elasticity, without any further assumptions. The beam is allowed to have large displacements and rotations, but small strain is assumed. The strain energy of the beam is computed making use of the constitutive law and the kinematical relations derived with the inclusion of geometrical nonlinearities and initial twist. Large displacements and rotations are allowed, but small strain is assumed. The Variational Asymptotic Method is used to minimize the energy functional, thereby reducing the cross section to a point on the reference line with appropriate properties, yielding a 1-D constitutive law. In this method as applied herein, the 2-D cross-sectional analysis is performed asymptotically by taking advantage of a material small parameter and two geometric small parameters. 3-D strain components are derived using kinematics and arranged as orders of the small parameters. Warping functions are obtained by the minimization of strain energy subject to certain set of constraints that renders the 1-D strain measures well-defined. Closed-form expressions are derived for the 3-D non-linear warping and stress fields. The model is capable of predicting interlaminar and transverse shear stresses accurately up to first order.

1 INTRODUCTION

Beams are flexible structures for which one of the dimensions is much large compared to the other two. To make use of the advantage of this geometric feature without loss of accuracy, one has to consider the behavior associated with the two dimensions eliminated in the model. In composite beams, there can be elastic couplings, which can further complicate the problem. The in- and out-of-plane warping can also be coupled. The stiffness model is strongly affected by these complications. Over the last decade, significant advances have been made in the area of analyzing composite beams[1].

In the area of beam modeling methods for rotor blade and flex-beam applications, significant amount of research work has been done. A detailed review of the various modeling techniques that was prevalent before 1990, was provided by Hodges[2]. In this review paper, rotor blade modeling is classified under two broad categories of theories: one, analytical approach and two, finite element (FEM) based approach. The analytical approach is characterized by closed-form calculation of the warping function and stiffness properties or by simple analytical approximations. This approach is appropriate and practicable only to simple geometric cross-sections like square or rectangular sections. The FEM approach overcomes this drawback of the analytical approach. It allows one to determine the warping function and elastic properties for any general cross-section geometry that can be modeled with standard two-dimensional (2-D) finite elements.

A classical theory captures extension, torsion and bending in two directions. After the variational asymptotic method (VAM) was developed by Berdichevsky (1976), the development of cross-sectional analyses for classical modeling of non-homogeneous, anisotropic beams are well understood. It is based on three-dimensional (3-D) nonlinear theory and it provides the basis for both the linear 2-D cross-sectional analysis and the one-dimensional (1-D) nonlinear equations. It has now been applied to prismatic beams in Hodges et al. (1992), initially curved and twisted beams in Cesnik et al. (1996) and Cesnik and Hodges (1997) and modeling of trapeze effect in Popescu and Hodges (1999).

The VAM was applied to the Timoshenko-like modeling of prismatic isotropic beams by Berdichevsky and Kvashnina (1976). Their work was extended to composite, prismatic beams in Popescu and Hodges (1999). VAM is a mathematical technique which can rigorously split the 3-D analysis of anisotropic beams into two problems: a 2-D analysis over the beam cross-section domain, which provides a compact form of the properties of the cross-sections, and a nonlinear 1-D analysis of the beam[3]. This work addresses closed, moderately-thick, pre-twisted laminated beams, which are popularly used as helicopter rotor flex-beams and wind turbine blades.

2 BEAM KINEMATICS

A typical beam cross-section could be described as a prescribed domain s with h as its characteristic size. The 2-D cross-sectional analysis is performed asymptotically by taking advantage of a material small parameter, namely the maximum allowable strain (ϵ), and two geometric small parameters, namely the ratio of maximum cross-sectional dimension to beam deformation wavelength ($\delta_h = h/l$) and the product of maximum cross-sectional dimension with the maximum pre-twist per unit length ($\delta_R = h * k_1$). The beam kinematics is derived for the pre-twisted beam in systematic steps as described below. Refer to Figure.1. The spatial position vector of any point on the undeformed

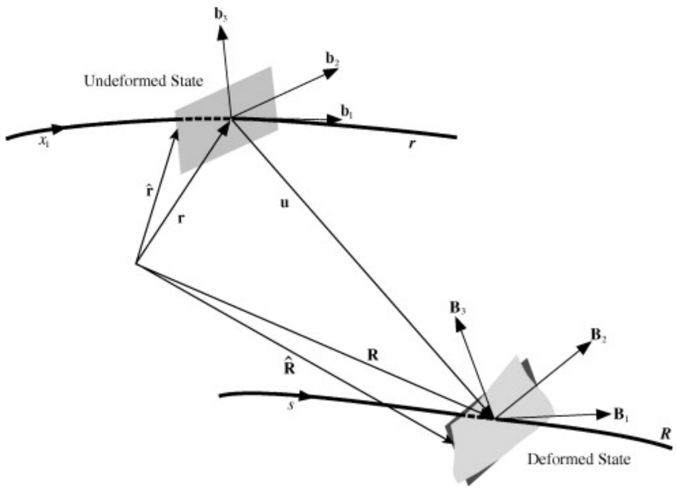

Figure 1: Beam deformation geometry

cross-section geometry is given by

$$\tilde{\mathbf{r}} = \mathbf{r}(x_1) + x_2\mathbf{b}_2(x_1) + x_3\mathbf{b}_3(x_1) \tag{1}$$

$$\mathbf{r}'(x_1) = \mathbf{b}_1(x_1) \tag{2}$$

where $\mathbf{b_i}$ cartesian system is defined in such a way that $\mathbf{b_1}(x_1)$ is always tangential to the beam reference curve. Pre-twist along the length-wise direction is introduced through the derivatives of the vectors $\mathbf{b_i}$, as follows.

$$\mathbf{b}_i'(x_1) = \mathbf{k} \times \mathbf{b}_i(x_1) \tag{3}$$

The position vector ($\tilde{\mathbf{R}}$) of an arbitrary material point in the deformed beam cross-section can be represented by

$$\tilde{\mathbf{R}} = \mathbf{R}(x_1) + x_2\mathbf{B}_2(x_1) + x_3\mathbf{B}_3(x_1) + \bar{w}_i(x_1, x_2, x_3)\mathbf{B}_i(x_1) \tag{4}$$

where $\mathbf{k} = k_1\mathbf{b}_1(x_1)$

The $\mathbf{B}_i$ is a right dextral triad defined for the deformed geometry and $\mathbf{w}_i$ are the components of warping. The one-dimensional strains $\gamma_{11}(x_1)$ and $2\gamma_{1\alpha}(x_1)$ are introduced through the derivative of the position vector of the deformed geometry.

$$\mathbf{R}'(x_1) = \mathbf{B}_1(x_1)(1 + \gamma_{11}(x_1)) + 2\gamma_{12}(x_1)\mathbf{B}_2(x_1) + 2\gamma_{13}(x_1)\mathbf{B}_3(x_1) \tag{5}$$

The one-dimensional curvature strains $\kappa_i(x_1)$ are introduced through the following relation

$$\mathbf{B}_{i'}(x_1) = \mathbf{K} \times \mathbf{b}_i(x_1) \tag{6}$$

where $\mathbf{K} = K_i \mathbf{B}_i(x_1)$ and $K_i = k_i + \kappa_i$

Eq. (4) is four times redundant because of the way warping is introduced. Hence, four appropriate constraints can be imposed on the displacement field to remove the redundancy. The four constraints applied here are as follows:

$$\ll \bar{w}_i(x_1, x_2, x_3) \gg = 0 \tag{7}$$

$$\ll \frac{\partial \bar{w}_3(x_1, x_2, x_3)}{\partial x_2} - \frac{\partial \bar{w}_2(x_1, x_2, x_3)}{\partial x_3} \gg = 0 \tag{8}$$

where the notation $\ll \cdot \gg$ implies integration over the reference cross-section. These equations ensure that the warping does not contribute to rigid body translations and rotations of the cross-section.

3 3-D FORMULATION

For moderately thick beams, the warping is small, the gradients of warping in the cross-sectional plane are of the same order, and the effect of local rotations is expected to be negligible. Danielson and Hodges' (1987) concept of decomposition of the rotation tensor is used to derive the Jauman-Biot-Cauchy strain components for small local rotation. The 3-D strain components for this case reduces to:

$$\Gamma_{ij} = \frac{1}{2}(A_{ij} + A_{ji}) - \delta_{ij} \tag{9}$$

where A is the deformation gradient tensor (defined in Eq. (10)), the components of which are arranged in mixed bases and given by Eq. (11).

$$\mathbf{A} = \mathbf{G}_i \mathbf{g}^i \tag{10}$$

$$A_{ij} = \mathbf{B}_i . \mathbf{A} . \mathbf{b}_j \tag{11}$$

The covariant ($\mathbf{g}_i$) and contravariant($\mathbf{g}^i$) base vectors for the undeformed geometry are defined by Eq. (12),(13). The covariant base vectors ($\mathbf{G}_i$) for the deformed geometry are defined by Eq. (14).

$$\mathbf{g}_i = \tilde{\mathbf{r}}_{,i} \tag{12}$$

$$\mathbf{g}^i = \frac{1}{\sqrt{2g}} e_{ijk} \mathbf{g}_j \times \mathbf{g}_k \tag{13}$$

$$\mathbf{G}_i = \tilde{\mathbf{R}}_{,i} \tag{14}$$

The analytical expression of $\mathbf{A}$ is obtained using the above-mentioned equations and its components are derived. Eq. (15)-(23) gives the complete expansion of all the components.

$$A_{11} = 1 + \mathrm{g}_{11} + x_3\kappa_2 - x_2\kappa_3 - \kappa_3\bar{w}_2 + \kappa_2\bar{w}_3 - k_1x_2\bar{w}_{1,3} + k_1x_3\bar{w}_{1,2} + \bar{w}_{1,1} \tag{15}$$

$$A_{12} = \bar{w}_{1,2} \tag{16}$$

$$A_{13} = \bar{w}_{1,3} \tag{17}$$

$$A_{21} = 2g_{12} + \kappa_3\bar{w}_1 - k_1\bar{w}_3 - \kappa_1\bar{w}_3 - k_1x_2\bar{w}_{2,3} + x_3(-\kappa_1 + k_1\bar{w}_{2,3}) + \bar{w}_{2,1} \tag{18}$$

$$A_{22} = 1 + \bar{w}_{2,1} \tag{19}$$

$$A_{23} = \bar{w}_{2,3} \tag{20}$$

$$A_{31} = 2g_{13} - \kappa_2\bar{w}_1 + k_1\bar{w}_2 + \kappa_1\bar{w}_2 + x_2(\kappa_1 - k_1\bar{w}_{3,3}) + k_1x_3\bar{w}_{3,2} + \bar{w}_{3,1} \tag{21}$$

$$A_{32} = \bar{w}_{3,2} \tag{22}$$

$$A_{33} = 1 + \bar{w}_{3,3} \tag{23}$$

The 3-D strain components derived using Eq. (9) arranged as orders of the small parameters in Eq. (24)-(29).

$$\Gamma_{11} = \underbrace{\gamma_{11} + x_3\kappa_2 - x_2\kappa_3}_{O(\varepsilon)} + \underbrace{\bar{w}_{1,1}}_{O(\varepsilon\delta_h)} - \kappa_3\bar{w}_2 + \kappa_2\bar{w}_3 - k_1x_2\bar{w}_{1,3} + k_1x_3\bar{w}_{1,2} \tag{24}$$

$$2\Gamma_{12} = \underbrace{\bar{w}_{1,2} - x_3\kappa_1}_{O(\varepsilon)} + \underbrace{2\gamma_{12} + \bar{w}_{2,1}}_{O(\varepsilon\delta_h)} + \kappa_3\bar{w}_1 - \kappa_1\bar{w}_3 - k_1\bar{w}_3 - k_1x_2\bar{w}_{2,3} + k_1x_3\bar{w}_{2,2} \tag{25}$$

$$2\Gamma_{13} = \underbrace{\bar{w}_{1,3} + x_2\kappa_1}_{O(\varepsilon)} + \underbrace{2\gamma_{13} + \bar{w}_{3,1}}_{O(\varepsilon\delta_h)} - \kappa_2\bar{w}_1 + \kappa_1\bar{w}_2 + k_1\bar{w}_2 - k_1x_2\bar{w}_{3,3} + k_1x_3\bar{w}_{3,2} \tag{26}$$

$$\Gamma_{22} = \underbrace{\bar{w}_{2,2}}_{O(\varepsilon)} \tag{27}$$

$$2\Gamma_{22} = \underbrace{\bar{w}_{2,3} + \bar{w}_{3,2}}_{O(\varepsilon)} \tag{28}$$

$$\Gamma_{33} = \underbrace{\bar{w}_{3,3}}_{O(\varepsilon)} \tag{29}$$

4 SOLUTION FOR WARPING

The strain energy density of the beam is given by Eq. (14).

$$U_{3D} = \frac{1}{2}\Gamma^T C \Gamma \tag{30}$$

$$U_{1D} = \iint_S U_{3D} dx_2 dx_3 \tag{31}$$

where C is the material constitutive matrix. Each lamina is assumed to be transversely isotropic, i.e., isotropy about a plane perpendicular to the fiber direction. For such a lamina, the matrix C is given by:

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{12} & C_{23} & C_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(C_{22} - C_{23}) & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{55} \end{bmatrix} \tag{32}$$

Warping functions are obtained by the minimization of strain energy (U_{1D}) subject to certain set of constraints given by Eq. (7),(8). Minimization yields a set of governing differential equations and boundary conditions. The governing equations are second order partial differential equations, while the boundary conditions are first order partial differential equations. Complete analytical solutions for these partial differential equations are quite impossible, so a semi-analytical approach is chosen. Eq. (33)-(35) gives the zeroth order expressions for warping which minimizes the zeroth order strain energy density.

$$\bar{w}_1^0 = -\kappa_1 x_2 x_3 \tag{33}$$

$$\begin{aligned} \bar{w}_2^0 = \; & \frac{C_{12}(C_{23}(H^2 - 12x_3^2)\kappa_3 + C_{33}(24x_2(\gamma_{11} + x_3\kappa_2) + B^2\kappa_3 - 12x_2^2\kappa_3))}{24(C_{23}^2 - C_{22}C_{33})} + \\ & \frac{C_{13}(-C_{22}(H^2 - 12x_3^2)\kappa_3 + C_{23}(-24x_2(\gamma_{11} + x_3\kappa_2) - B^2\kappa_3 + 12x_2^2\kappa_3)) - B^2\kappa_3 + 12x_2^2\kappa_3))}{24(C_{23}^2 - C_{22}C_{33})} \end{aligned} \tag{34}$$

$$\begin{aligned} \bar{w}_3^0 = \; & \frac{C_{12}(C_{23}(B^2 - 12x_2^2)\kappa_2 + C_{23}(-24x_3(\gamma_{11} - x_2\kappa_3) + H^2\kappa_2 - 12x_3^2\kappa_2))}{24(C_{23}^2 - C_{22}C_{33})} + \\ & \frac{C_{13}(-C_{23}(B^2 - 12x_2^2)\kappa_2 + C_{22}(24x_3(\gamma_{11} - x_2\kappa_3) - H^2\kappa_2 + 12x_3^2\kappa_2)) - H^2\kappa_2 + 12x_3^2\kappa_2))}{24(C_{23}^2 - C_{22}C_{33})} \end{aligned} \tag{35}$$

In order to find the next approximation to the 1-D energy density, the previous classical approximation of the warping is perturbed.

5 CROSS-SECTIONAL STIFFNESS MATRIX

The zeroth-order 3-D warping field thus yielded is then used to integrate the 3-D strain energy density over the cross-section, resulting in the 1-D strain energy density which in turn helps identify the corresponding cross-sectional stiffness matrix (S^0), as follows:

$$S^0 = \begin{bmatrix} BHK_0 & 0 & 0 & 0 \\ 0 & 0.33BH^3K_0 & 0 & 0 \\ 0 & 0 & (BH^3/12)K_0 & 0 \\ 0 & 0 & 0 & (B^3H/12)K_0 \end{bmatrix} \tag{36}$$

where $K_0 = \frac{C_{11}+(C_{13}^2C_{22}-2C_{12}C_{13}C_{23}+C_{12}^2C_{33})}{C_{23}^2-C_{22}C_{33}}$, B is the breadth and H is the height of the rectangular cross-section.

For an isotropic material, the zeroth-order solution is equivalent to the classical Euler-Bernoulli beam theory, as K_0 boils down to its Young's modulus (E). The 1-D transverse shear strain terms ($2\gamma_{12}(x_1)$ and $2\gamma_{13}(x_1)$) figure in the 3-D strains of first-order and the first-order warping component is computed by minimisation of the asymptotically accurate first-order strain energy[4].The resulting first-order cross-sectional stiffness matrix (S^1) is a 6 x 6 matrix with transverse shear stress components, as in the case of the Timoshenko beam theory, which however is asymptotically incorrect.

$$S^1 = \begin{bmatrix} BHK_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & S_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.33BH^3K_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & (BH^3/12)K_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & (B^3H/12)K_1 \end{bmatrix} \tag{37}$$

where $K_1 = K_0$, $S_{22} = 2.133BHC_{66} + \frac{3.556H^3C_{66}^2}{BC_{55}} + \frac{0.8H^5C_{66}^3}{B^3C_{55}^2}$ and
$S_{33} = 2.133BHC_{55} + \frac{0.8B^5C_{55}^3}{H^3C_{66}^2} + \frac{3.556B^3C_{55}^2}{HC_{66}}$

This model can be further expanded for an n-ply laminate, with different ply orientations, to explore the interlaminar stresses at various locations along the length of the beam. In that case, the individual strain energies of each plies are summed up to obtain the strain energy density of the laminate. Subsequently, warping functions are evaluated for each plies and then used to derive the cross-sectional stiffness matrix for the laminate. The results for a simple two-ply laminate [0/0] is given below:

$$\bar{S} = \begin{bmatrix} B(K[1]t_1 + K[2]t_2) & 0 & B(-K[1]t_1^2 + K[2]t_2^2) & 0 \\ 0 & 0.33B(C_{66}[1]t_1^3 + C_{66}[2]t_2^3) & 0 & 0 \\ B(-K[1]t_1^2 + K[2]t_2^2) & 0 & (13B/12)(K[1]t_1^3 + K[2]t_2^3) & 0 \\ 0 & 0 & 0 & (B^3/12)(K[1]t_1 + K[2]t_2) \end{bmatrix}$$

where $K[1] = \frac{C_{11}[1]+(C_{13}^2[1]C_{22}[1]-2C_{12}[1]C_{13}[1]C_{23}[1]+C_{12}^2[1]C_{33}[1])}{C_{23}^2[1]-C_{22}[1]C_{33}[1]}$ and

$K[2] = \frac{C_{11}[2]+(C_{13}^2[2]C_{22}[2]-2C_{12}[2]C_{13}[2]C_{23}[2]+C_{12}^2[2]C_{33}[2])}{C_{23}^2[2]-C_{22}[2]C_{33}[2]}$. [1] and [2] represents bottom and top ply respectively.

6 CONCLUSIONS

A generalized cross-sectional modeling approach for moderately thick anisotropic beams with pre-twist has been presented in this paper. A Tiimoshenko-like model is generated with 6 x 6 cross-sectional stiffness matrix along with the solution of the warping for a pre-twisted laminated composite beam. Hence the model yields an accurate semi-analytical solution for thick beams with rectangular cross sections. The main contributions of this work are:

- Semi-analytical expressions for transverse shear strains can be derived asymptotically for moderately thick laminates with rectangular cross-sections.

- Cross-sectional stiffness matrix similar to a Timoshenko-like model can be obtained for any general pre-twisted beams, which can be used to further analyze the non-linear 1-D beam reference curve. Helicopter rotors consisting of composite tapered flex beams are subjected to high cycles of fatigue and are prone to delamination at the ply drop-off regions. Such complicated structures can be analysed using this model and closed-form solutions can be obtained to have a better understanding of thick beams.

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