11th World Congress on Computational Mechanics (WCCM XI)
5th European Conference on Computational Mechanics (ECCM V)
6th European Conference on Computational Fluid Dynamics (ECFD VI)
E. Oñate, J. Oliver and A. Huerta (Eds)

MIXED FINITE ELEMENT MODEL IMPLEMENTATION FOR A PETROLEUM RESERVOIR SIMULATION

Carlos M. Osorio and Omar D. López

Departamento de Ingeniería Mecánica
Universidad de los Andes
Edificio Mario Laserna Cra 1 Este No 19A - 40, Bogotá, Colombia
e-mail: cm.osorio163@uniandes.edu.co, od.lopez20@uniandes.edu.co

Key words: Reservoir simulation, two-phase flow, MFEM, LSFEM, MMOCAA

Abstract. In this work a simulation of a petroleum reservoir using FreeFEM++ v 3.21-1 is performed with a two phase (water and oil) incompressible and immiscible flow. The rock is assumed to be porous and immovable. For the pressure equation, Mixed Finite Elements such as Taylor Hood and Raviart Thomas, which are convergent not only for pressure but also for velocity were used. Different kinds of boundary conditions such as Velocity (Neumann) and Bottom Hole Pressure (Robin) were used. Local mass conservation is improved using an additional term in the weak formulation proposed by Masud et al. related to Least Squares Finite Element Method (LSFEM) [17]. For the Saturation equation, which is hyperbolic in its nature, two methods are used: A Modified Method of Characteristics with Adjusted Advection (MMOCAA) [9], which improves the global mass conservation compared to the traditional MMOC. The other one is the Galerkin Finite Element Method using a non-linear artificial viscosity [8]. Two numerical tests were performed: one using a uniform permeability and the other using layer 85 of the Model 2 from SPE10 [7] as geostatistical permeability. This Upper Ness formation is a challenge for the simulations due to its high heterogeneity and bimodality. Watercut curves and instantaneous fields showing evolution of variables (pressure, velocity and saturation of water) are presented as numerical results of this work. More similarity is achieved to the watercut curve show in reference [14] using the artificial viscosity, although both methods (MMOCAA and Galerkin) respect the global mass conservation.

1 INTRODUCTION

The flux problem or Darcy's law applied to multiphase media has been particularly studied from a computational point of view mainly because of its numerical instability. Petroleum reservoirs are a clear physical example of a Darcy problem: there is a porous solid medium, which is the unmovable rock, and the liquid components and phases flowing through it. The incompressible flow through porous media present interesting numerical difficulties that are still of main concern in the scientific community. This phenomenon consists of two equations: one is the 'Pressure' or 'Darcy' equation that has elliptic nature and relates pressure and velocity. The other one is the 'saturation equation', which can be hyperbolic or parabolic, depending on the variables handled. In the finite element technique for Darcy's equation, this problem is well known due to the need of using the mixed formulation instead of the Galerkin, which handles the boundary conditions and type of elements in a different manner. Besides FEM, several authors such as Gunzburger, Bochev and Hughes have referenced the Least Squares FEM (LSFEM) as an ideal method for mass conservation, and also for reducing instabilities. This method gives additional terms to the weak formulation. On the other hand, saturation equation has known numerical stability problems, due to the assimetry of the advective operator. Langrangian methods such as MOC and its derivates are used to remove that operator. Another way to solve it propose an artificial viscosity which makes a parabolic or convective-diffusive equation.

Raviart and Thomas [21] proposed a suitable element for vector variables such as velocity, which are different from Lagrange finite elements used for the Galerkin formulation. This element propose continuity of velocity normal components in all the element edges. Then, Brezzi , Douglas and Marini raised [4] a new element, similar to Raviart Thomas, but with fewer degrees of freedom for the same order of convergence. A common problem when using the primal formulation, which only has the pressure as a variable, is that when taking the gradient of this, and multiplying by the mobility to find the velocity, there is a loss of accuracy in the results and so the mass conservation of the numerical method is not guaranteed [17]. Authors such as Hughes, Bochev and Gunzburger [3, 2, 17] suggest the usage of weak formulations based on least squares finite elements (LSFEM) for this purpose.

Saturation equation can have different natures depending on the behavior of the relative permeabilities. In the Buckley Leverett equation [5], the relative permeability of each phase depends quadratically on the saturation thereof . This equation has a non - linear behavior and stability has been studied by different authors [13]. As any hyperbolic equation, it is not possible to solve the saturation equation using the finite element method directly, so methods such as the characteristics (MOC) [10] reduces the partial time derivative and the advective operator into a single material time derivative. When this method is combined with the Galerkin finite elements, the possibilities of solving convective - diffusive equations arise: this is the Modified method of characteristics (MMOC) [15]. However, MMOC has deficiencies related to mass conservation, reported around 10% in geostatistical reservoirs [6, 9]. That is why an ad-hoc adjustment to advection in the method of characteristics was proposed, which reaches the mass conservation at least globally. This method is called the Modified Method of Characteristics with Adjusted Advection (MMOCAA) [9]. Alongside these Lagrangian methods, the Artificial viscosity method emerges as the principal Eulerian approach. Gerard and Pasquetti [11] raised an artificial viscosity for 1D saturation equation, which increases depending on the residue of the differential equation. This same viscosity was used by Chueh et al [8] for a 2D h-adaptive problem, refining the mesh in areas where the saturation gradient was high while the time step was also adaptive depending on the Courant number. Methods such as SUPG and capture of high gradients [20] modify the weight function for matrix symmetrization, however, they can also be considered artificial viscosity methods that add terms of this kind to the weak formulation.

This paper summarizes the implementation and use of mixed finite elements on stable weak formulations for simulation of reservoirs, specifically for the Darcy equation, and two different methods used for the saturation equation that conserve mass in a global approach. These techniques are especially implemented for the incompressible two-phase model.

2 GOVERNING EQUATIONS AND SIMPLIFICATIONS

2.1 Two phase model

The water-oil two phase model is used in reference [19]. This model permits incompressibility, no vacuum and no mass transfer between phases. Equations 1 and 2 relate to the non-wetting (oil) and wetting (water) phases respectively.

.

$$\nabla \cdot \left[\frac{\rho_o K k_{ro}}{\mu_o} (\nabla p_o - \rho_o \mathbf{g}) \right] + q_o = \frac{\partial(\phi \rho_o S_o)}{\partial t} \tag{1}$$

$$\nabla \cdot \left[\frac{\rho_w K k_{rw}}{\mu_w} (\nabla p_w - \rho_w \mathbf{g}) \right] + q_w = \frac{\partial(\phi \rho_w S_w)}{\partial t} \tag{2}$$

Although there are four unknowns (two saturations and two pressures) there are only two equations. As both components fill the pores completely, the addition of both saturations must be one. The remaining equation 3 is given by a relation between two pressures, the capillary pressure:

$$S_o + S_w = 1 \qquad\qquad p_c(S_w) = p_o - p_w \tag{3}$$

2.2 Assumptions and simplifications

In order to simplify eqs. 1, 2 and 3, some assumptions are taken: no gravity ($g = 0$), no capillary pressure ($p_o = p_w$), incompressibility ($\partial\rho/\partial t = 0$) and steady-state porosity ($\partial\phi/\partial t = 0$). There are also some additional definitions, known as mobility (λ) and relative mobility (f):

$$\lambda_o = \frac{k_{ro}}{\mu_o} \qquad\qquad \lambda_w = \frac{k_{rw}}{\mu_w} \qquad\qquad \lambda_t = \lambda_o + \lambda_w$$

$$\mathbf{v} = \mathbf{v}_o + \mathbf{v}_w \qquad f_w = \frac{\lambda_w}{\lambda_t} \qquad f_o = \frac{\lambda_o}{\lambda_t} \tag{4}$$

Also, it is assumed that each production Q is proportional to its relative mobility [9]

$$Q_w = f_w Q, \quad Q_o = (1 - f_w)Q \tag{5}$$

2.3 Final equations

Given the previous assumptions and simplifications, the divergence of the velocity field (conservation of mass) is given by:

$$\mathbf{v} = -K\lambda_t \nabla p \qquad \nabla \cdot \mathbf{v} = Q \tag{6}$$

The saturation equation can be written in two ways: the divergence form and the non-divergence form. These equations come from eq. 2.

$$\phi \frac{\partial S_w}{\partial t} + \nabla \cdot (f_w \mathbf{v}) = Q_i - f_w Q_p \tag{7}$$

$$\phi \frac{\partial S_w}{\partial t} + \frac{df_w}{dS_w} \mathbf{v} \cdot \nabla S_w = (1 - f_w) Q_i \tag{8}$$

The divergence form is useful for Galerkin methods, while the non-divergence form is used for Lagrangian methods. All the boundaries of the domain are assumed impermeable ($\mathbf{v} \cdot \mathbf{n} = 0$), so that the injections and extractions are simulated as sources and sinks.

3 NUMERICAL METHOD

3.1 Weak formulation for Darcy's equation

The original mixed formulation given for eq. 6 must include the boundary terms [16] (Nitsche's condition) and a term given by Hughes et al [17]. This term is based in the LSFEM formulation and improves the local mass conservation for the pressure equation. Taylor Hood finite elements were used, presenting a linear interpolation for pressure and a quadratic interpolation for velocity. Equation 9 shows the final weak form of eq. 6.

$$\int_\Omega [\lambda_t K]^{-1} \mathbf{v} \cdot \delta \mathbf{v} - \int_\Omega p \nabla \cdot \delta \mathbf{v} + \int_\Omega \alpha_{div} [2\lambda_t K]^{-1} h^2 (\nabla \cdot \delta \mathbf{v})(\nabla \cdot \mathbf{v} - Q) + \int_{\Gamma_i} \alpha_i (\mathbf{v} \cdot \mathbf{n})(\delta \mathbf{v} \cdot \mathbf{n}) = 0$$

$$\int_\Omega \delta p \nabla \cdot \mathbf{v} - \int_\Omega \delta p Q = 0 \qquad \forall \delta \mathbf{v} \in H_{div}(\Omega) \qquad \forall \delta p \in L^2(\Omega) \tag{9}$$

3.2 Modified Method of Characteristics (MMOC)

The characteristics method computes the position of a particle based in the velocity field estimated from the pressure equation. As the MMOC uses the non-divergence form (eq. 8) a modified velocity field $\mathbf{v}_m = \frac{\mathbf{v}}{\phi}\frac{df_w}{dS_w}$ must be determined. The MMOC has global mass conservation problems, reported by several authors [6]. This issue forces a different approach that respects the mass conservation, at least locally.

3.3 MMOCAA

The Modified Method of Characteristics with Adjusted Advection (MMOCAA) [9, 12] gives a saturation range in which the answer is selected from a linear interpolation, related to the mass conservation:

$$\mathbf{v}_m^+ = (1+\gamma\Delta t)\mathbf{v}_m \qquad \mathbf{v}_m^- = (1-\gamma\Delta t)\mathbf{v}_m$$

$$S_w^+ = convect(\mathbf{v}_m^+, -dt, S_w) \qquad S_w^- = convect(\mathbf{v}_m^-, -dt, S_w) \tag{10}$$

Mass conservation interpolation requires to find some variables related:

$$R^+ = \int_\Omega \left[\phi S_w^+\right] \qquad R^- = \int_\Omega \left[\phi S_w^-\right] \qquad R = \int_\Omega \left[\phi S_{wo} + \Delta t\left(f_w Q_p\right)\right] \tag{11}$$

$$\theta = \frac{R-R^-}{R^+-R^-} \qquad \hat{S}_w = \theta S_w^- + (1-\theta)RS_w^+ \tag{12}$$

Finally, the convected saturation is given by:

$$S_{w,t+1}(\mathbf{x}_{t+1}, t+1) = \hat{S}_{w,t}(\mathbf{x}_t, t) + \Delta t\frac{(1-f_w)Q_i}{\phi} \tag{13}$$

3.4 Artificial viscosity

Chueh et al [8] proposes an artificial viscosity for the saturation equation, becoming into a convective-diffusive relation. Prior to the artificial viscosity, normalization constant and residual are given by eq. 14:

$$c(\mathbf{v}, S_w) = c_R \left\|\mathbf{v}\frac{df_w}{dS_w}\right\|_{L^\infty(\Omega)} \left(\max_\Omega S_w - \min_\Omega S_w\right) diam(\Omega)$$

$$Res(S) = \left(\phi\frac{\partial S_w}{\partial t} + \mathbf{v}\cdot\nabla f_w - (1-f_w)Q_i\right) S_w^{\alpha-1} \tag{14}$$

The artificial viscosity field is given by:

$$\nu_c = \beta \left\|\mathbf{v}\max\left\{\frac{df_w}{dS_w}, 1\right\}\right\|_{L^\infty(\Omega_e)} \min\left\{h, h^\alpha \frac{\|Res(S)\|_{L^\infty(\Omega_e)}}{c\left(\mathbf{v}, S_w\right)}\right\} \tag{15}$$

Additionally, high viscosity values ν_o were strategically located in areas of the domain where saturation outliers outside [0,1] were obtained, and therefore numerical instabilities were observed. Finally, the weak formulation used for this problem is given by:

$$\int_\Omega \left[\phi W \frac{\partial S_w}{\partial t} - f_w \mathbf{v} \cdot \nabla W + (\nu_c + \nu_o) \nabla S_w \cdot \nabla W - W Q_i + W f_w Q_p \right] = 0 \qquad (16)$$

4 ALGORITHM

4.1 Pore Volume Injected

Pore Volume Injected (PVI) is the most common way of time adimensionalization in reservoir simulation. It gives the portion of water injected compared to the total pore volume in the reservoir (Equation 17). Pore Volume Produced is an additional definition used for global mass conservation.

$$PVI = \left[\int_\Omega \phi d\Omega \right]^{-1} \int^t \int_\Omega Q_i dt \qquad PVP = \int^t \int_\Omega f_w Q_p dt \qquad (17)$$

4.2 IMPES method

The IMPES method, which stands for Implicit Pressure Explicit Saturation, solves the pressure equation weak formulation given by eq. 9 first. Then uses the resultant velocity field to solve the saturation equation 13 or 16, depending on the method used. The UMFPACK direct solver is used for the resultant linear system in each step for the pressure equation. On the other hand, the saturation equation includes a non-linear term f_w, which is quadratic in the Buckley Leverett case. So for that, a Gauss Seidel method, which is the simplest nonlinear solving technique, must be implemented. In order to ensure numerical stability, he CFL condition must be always less than 1.

5 NUMERICAL EXAMPLE

5.1 SPE Geometry and boundary conditions

The data given by the Tenth SPE Comparative Solution Project, also called SPE10, which simulate the water flow in two models with several upscaling techniques is commonly used as a benchmark model for petroleum reservoir simulation [7]. For the present study, only the geostatistical permeability is used. Inclusion of a geostatistical porosity suppose an additional problem related to the saturation equation, and also, most of the authors prefer it to be uniform [9, 1, 14]. Its dimensions are 1200 ft x 2200 ft x 170 ft. The fine scale has 60 x 220 x 85 cells. So each fine cell is 20 ft x 10 ft x 2 ft. The model has 85 layers, given by the number of cells in the z direction. The first 35 top layers represent the Tarbert formation, while the bottom 50 layers are of the Upper Ness formation. Tarbert formation is relatively uniform, instead of the Upper Ness which has a fluvial nature, that means the presence of flux channels with higher permeabilities and porosities. In other words, while the Tarbert formation unimodal, the Upper Ness formation is bimodal. Further details can be found on [18]. The geometry is based in layer 85 of the SPE10, and its sides are 220 units x 60 units, taking each cell as an 'unit', and therefore using a different geometry from the original one. This strategy was also used by Jiang & Mishev [14]

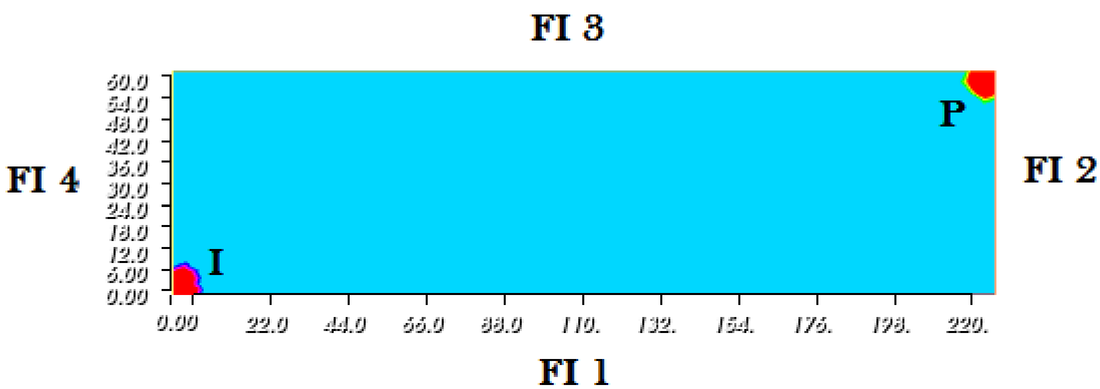

Figure 1: Geometry of geostatistical reservoir

The reservoir has an injector in point (0,0), a producer in the point (220,6) and it is enclosed in a rectangular impermeable boundary (see figure 1).The used permeability is highly heterogeneous (Upper Ness), as it is evident in figure 2. Coarse mesh and fine mesh have 110 x 30 and 220 x 60 elements respectively. They use also $\Delta t = 0.05$ and $\Delta t = 0.025$ respectively in order to respect CFL condition. Porosity is 1 for the entire domain. Tables 1 and 2 summarize the physical and numerical parameters used in the numerical test.

Symbol	**Parameter**	**Value**
μ_w/μ_o	Viscosity ratio	1/10
S_{wc}, S_{or}	Critical saturations	$S_{wc} = S_{or} = 0$
S_0	Initial saturation	$S_0 = S_{wc} = 0$
S_m	Modified saturation	$S_m = (S - S_{wc})/(1 - S_{wc} - S_{or})$
k_{rw}	Relative permeability - water	$k_{rw} = S_m^2$
k_{ro}	Relative permeability - oil	$k_{ro} = (1 - S_m)^2$
K	Absolute permeability	SPE10-85
ϕ	Porosity	1

Table 1: Adimensional physical parameters for geostatistical reservoir

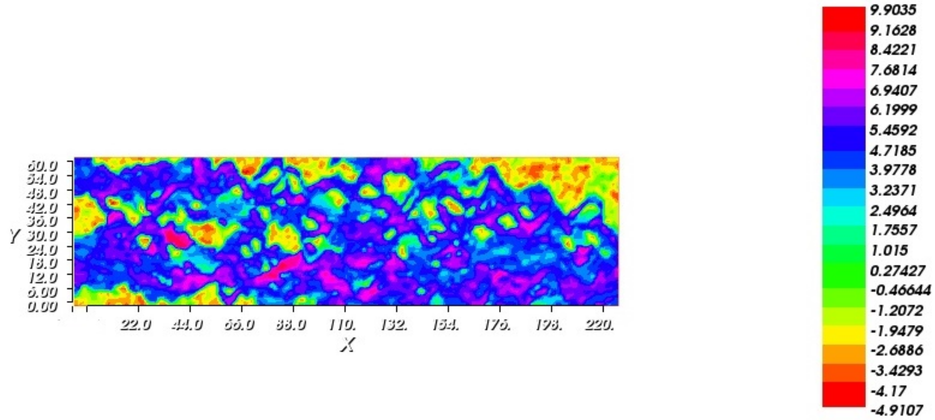

Figure 2: Natural logaritm of permeability used for geostatistical reservoir

Symbol	Parameter	Value
α	Stabilization exponent AV	1
β	Stabilization constant AV	0.8
c_R	Normalization parameter AV	1
Δt	Timestep	$0.05, 0.025$
γ	MMOCAA parameter	20
α_{div}	Divergence error parameter	2
ε	Neumann BC parameter	10^5
it	Gauss Seidel max. iteration	5
r_h	Assumed well radius	8

Table 2: Numerical parameters for geostatistical reservoir

5.2 Watercut

This variable is given by the following relation:

$$watercut = \frac{\int_\Omega Q_p f_w}{\int_\Omega Q_p}$$

Figure 3 (left) shows the watercut evolution, and two phases are clearly evidenced: one in which the Watercut is zero, that means the water has not reached the producer yet. The other phase is when the water begins to flow across the producer. It can be observed that the MMOC (purple line) has severe global mass conservation inconsistencies. The MMOCAA values for watercut (green line) take away from the author values (red line) [14]. The MMOCAA changes the characteristic lines, and then the resultant curve is proportional to the given curve for the non-conservative MMOC method. The way this method changes the characteristic lines is related to an homogenous theta for the entire

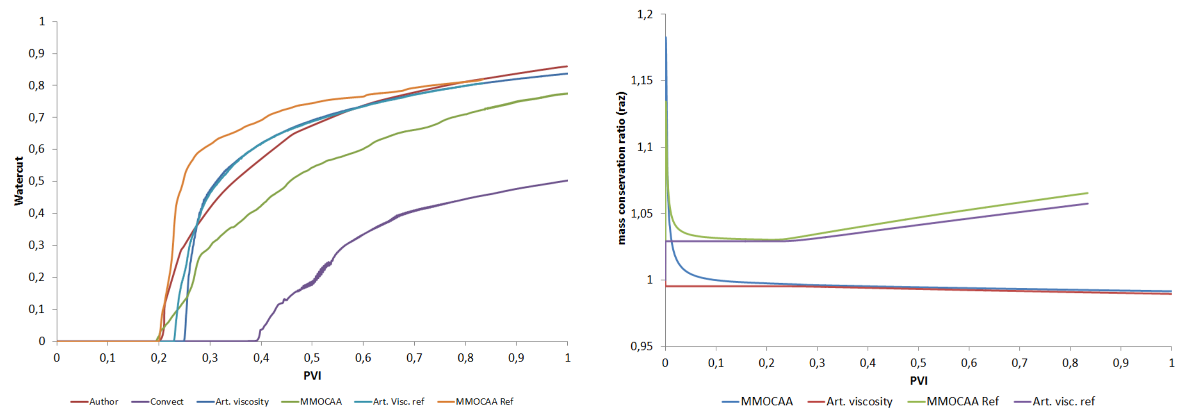

Figure 3: Left: Watercut for several simulations. Right: Global mass conservation ratio for several simulations

domain, that means the local mass conservation is not achieved due to the average value used.

The Galerkin Finite Element Method, headed by the artificial viscosity in this case (dark blue line), presents more similarity to the author's curve. This is because the finite element principle automatically respects the mass conservation in each element. All these results were obtained using the coarse mesh. The mesh refinement influences positively the artificial viscosity results (clear blue line), moving the instant when the water reaches the producer closer to the author's curve. In the MMOCAA (orange line) there is also a major proximity to Mishev's data, however, the watercut is always greater than the author's.

5.3 Global mass conservation

The global mass conservation must be verified in two cases: In the first one, there's only injected water in the reservoir, and this water have not reached the producer jet. In that case only a comparison between injected and present water is needed. After water reaches the producer, we need to take into account also the PVP, or pore volume produced:

$$raz = \frac{PVI - PVP}{\int_\Omega \phi(S_w - S_{w0})} \qquad PVP = \int^t \int_\Omega f_w Q_p dt$$

There is a good fulfillment of global mass conservation in coarse mesh, as shown in figure 3 (right). There's only a small singularity close to 117% at the beginning of MMOCAA method, but as it is expected, the numbers compared in that instant are very small so this singularity is not dangerous at all. For both methods and coarse mesh, the raz variable constantly decreases to 99% approx.

5.4 Local mass conservation for Darcy's equation

Local mass conservation for elliptic Darcy equations is one of the most studied problems nowadays by the finite element scientific community. The divergence control term used by Masud & Hughes improves this variable substantially. However it still has some issues that rise when simulating geostatistical reservoirs and specified productions. The term α_{div} and a smooth permeability positively influence the local mass conservation. Another important finding in this local mass conservation problem, was that the non-geostatistical problem, which uses uniform permeability, prefers a high α_{div} that will reduce the difference as it increases (see complete document [18]), but the inclusion of a non-smooth permeability adds a problem for the local mass error, which increases instead of decreasing when α_{div} is high (shown in figure 4), so for that case, we should find an intermediate α_{div} which diminishes this error, instead of increasing it. By the other way, we have to take into account also that for the geostatistical problem, production and total impermeability boundary conditions are used, instead of velocity boundaries, where the divergence in all the domain is zero. This forces the divergence to be different from zero in the production and injector wells. It was also found that the local mass conservation remains the same across time and it also gets better when the mesh is refined. A further explanation about local mass conservation, and its behavior depending of α_{div}, geostatistics and mesh refinement can be found on [18]

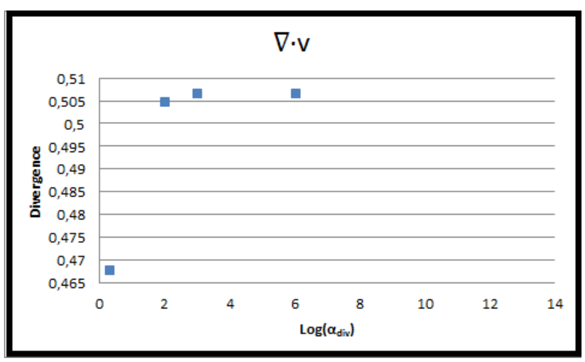

Figure 4: Divergence error vs. log (α_{div}) for geostatistical problem

6 CONCLUSIONS

In this work a mixed finite element algorithm for petroleum reservoir simulations was proposed and implemented. The use of mixed finite elements seems very attractive, not only for the simulation of this kind of problem, but also for other uses such as drying processes. This features, combined with the local mass conservation improvement given by LSFEM in Darcy's equation, lead to an oustanding method for reservoir simulation. Principally, the geostatistical case needs an intermediate α_{div} (see figure 4) which minimizes the divergence of the velocity field (that should be zero in non-production points), in a heuristic way, or using an optimization algorithm as well. The MMOCAA method has convergence problems, at it is observed in figure 3 for the watercut, evident in the difference between coarse and fine mesh curves for this variable. The artificial viscosity, in its way, get closer to the author's curve as the mesh is refined, and both coarse and fine curves are very close. This is due to the lagrangian nature of MMOCAA method. Although the pressure equation is quite well understood, the saturation equation is still a challenge in the numerical methods field.

Global mass conservation was quite accomplished, as well as the watercut curve (see figure 3). An extremely heterogeneous and challeging permeability was used, and the robustness of this method was verified. It is expected that for the near future, the advantages of both Galerkin and Characteristics method combine, and in that way, this method can compete with Finite Volume approaches.

REFERENCES

[1] Jørg E. Aarnes, Vegard Kippe, and KnutAndreas Lie. Mixed multiscale finite elements and streamline methods for reservoir simulation of large geomodels. *Advances in Water Resources*, 28(3):257–261, 2005.

[2] P. Bochev and M. Gunzburger. *Least Squares Finite Element Methods*. Springer, Berlin, 2009.

[3] P. B. Bochev and M. D. Gunzburger. A locally conservative least-squares method for darcy flows. *Communication in Numerical Methods in Engineering*, Volume 24 Issue 2:97–110, 2008.

[4] Franco Brezzi, Jim Douglas Jr., and L. D. Marini. Two families of mixed finite elements for second order elliptic problems. *Numerische Mathematik*, Volume 47 Issue 2:217–235, 1985.

[5] S.E. Buckley and M.C. Leverett. Mechanism of fluid displacement in sands. *Transactions of the AIME*, Volume 146, Number 1:107 – 116, 1942.

[6] Zhangxin Chen, Guanren Huan, and Yuanle Ma. *Computational Methods for Multiphase Flows in Porous Media*. Society for Industrial and Applied Mathematics, 2006.

[7] M.A. Christie, Heriot-Watt U, and M.J. Blunt. Tenth spe comparative solution project: A comparison of upscaling techniques. *SPE Reservoir Simulation Symposium*, 10:308–317, 2001.

[8] Chih-Che Chueh, Ned Djilali, and Wolfgang Bangerth. An h adaptive operator splitting method for two phase flow in 3d heterogeneous media. *SIAM Journal of Scientific Computing*, 45:B149–B175, 2013.

[9] Jr. Douglas, Jim, Frederico Furtado, and Felipe Pereira. On the numerical simulation of waterflooding of heterogeneous petroleum reservoirs. *Computational Geosciences*, 1(2):155–190, 1997.

[10] A.O. Garder Jr., D.W. Peaceman, and A.L. Pozzi Jr. Numerical calculation of multidimensional miscible displacement by the method of characteristics. *SPE Journal*, Volume 4, Number 1:26 – 36, 1964.

[11] Jean-Luc Guermond and Richard Pasquetti. Entropy-based nonlinear viscosity for fourier approximations of conservation laws. *C. R. Acad. Sci. Paris*, Ser I. 346:801–806, 2008.

[12] Chieh-Sen Huang. Convergence analysis of a mass-conserving approximation of immiscible displacement in porous media by mixed finite elements and a modified method of characteristics with adjusted advection. *Computational Geosciences*, 4(2):165–184, 2000.

[13] Patrick Jenny, Hamdi A. Tchelepi, and Seong H. Lee. Unconditionally convergent nonlinear solver for hyperbolic conservation laws with s-shaped flux functions. *Journal of Computational Physics*, 228(20):7497 – 7512, 2009.

[14] L. Jiang and I. D. Mishev. Mixed multiscale finite volume methods for elliptic problems in two-phase flow simulations. ArXiv e-prints, 2011.

[15] Jr. Jim Douglas and Thomas F. Russell. Numerical methods for convection-dominated diffusion problems based on combining the method of characteristics with finite element or finite difference procedures. *SIAM Journal on Numerical Analysis*, Vol. 19, No. 5:pp. 871–885, 1982.

[16] Juho Könnö, Dominik Schötzauy, and Rolf Stenberg. Mixed finite element methods for problems with robin boundary conditions. *SIAM Journal of Numerical Analysis*, 49:285–308., 2011.

[17] Arif Masud and Thomas J.R. Hughes. A stabilized mixed finite element method for darcy flow. *Computer methods in applied mechanics and engineering*, 191:4341–4370, 2002.

[18] C. M. Osorio and O. D. Lopez. Modelo computacional de elementos finitos mixtos aplicados a la solución numérica de la ecuación de darcy. Master's thesis, Universidad de los Andes, 2014.

[19] Donald Peaceman. *Fundamentals of Numerical Reservoir Simulation.* Elsevier Scientific Publishing Company, 1977.

[20] G. O. Porras, F. Couture, and M. Roques. A convection-diffusion model for the convective drying of a shrinking medium composed of a binary liquid. *Drying Technology*, 25:1215–1227, 2007.

[21] P. A. Raviart and J. M. Thomas. A mixed finite element method for 2nd order elliptic problems. *Mathematical Aspects of Finite Element Methods*, 606:292–315, 1977.