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A MIXED CUBIC ZIGZAG MODEL FOR MULTILAYERED COMPOSITE AND SANDWICH PLATES INCLUDING TRANSVERSE NORMAL DEFORMABILITY

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Abstract. A new Mixed Cubic Zigzag Theory ($CZT^{(m)}$) has been developed via the Reissner Mixed Variational Theorem. The assumed kinematic field postulates an in-plane displacement components piecewise cubic along the thickness and a smeared parabolic through-the-thickness distribution for the transverse displacement. The assumed transverse shear stresses profile derives from integration of the three-dimensional equilibrium equations whereas the normal stress pattern is assumed smeared cubic along the laminate thickness. The entire formulation is here developed and the governing equations derived are used to solve the bending problem of a rectangular simply-supported cross-ply plate subjected to a bi-sinusoidal transverse load. In order to assess the predictive capabilities of the $CZT^{(m)}$ model, results are compared with the exact three-dimensional Elasticity solution.

1 INTRODUCTION

The increasing use of laminated composite and sandwich materials as primary geometrically complex load-bearing components in the structures of modern civilian and military aircraft, requires tools able to accurately predict the global and, above all, local response of these components. In presence of geometrical singularity, like as an hole or only a free edge, the state of stress becomes three-dimensional. The onset of transverse shear and normal stresses represents a severe working condition for a laminates, since they are responsible for the delamination and debonding. Thus, a reliable design of these components demands accurate evaluation of the six stress tensor components.

During the years, several Equivalent Single Layer (ESL) models, wherein the multilayer laminate is replaced with an equivalent single layer, have been developed [1,2]. These models result not accurate in predicting the global and, above all, local response of laminates and sandwiches. High level of accuracy is ensured by Layer-Wise (LW) models [1,2] at the expense of the computational cost. On the contrary, the zigzag class of structural theories, pioneered by Di Sciuva [3], are characterized by an accuracy comparable to that of the LW approaches and a computational cost similar to the ESL models.

Recently, Tessler et al. [4] formulated the Refined Zigzag Theory (RZT), wherein the First-Order Shear Deformation Theory kinematic is enriched by adding a novel set of piece-wise linear functions, thus resulting in a seven-kinematic variables model. The RZT accuracy has been already assessed on several problems concerning the bending, free vibration and buckling of moderately thick multilayered composite and sandwich plates [4,5] In order to improve the transverse shear stresses description, lately Iurlaro et al. [6] have developed a mixed RZT model ($RZT^{(m)}$), via the Reissner Mixed Variational Theorem [7], wherein the assumed transverse shear stresses are derived with the aid of the three-dimensional equilibrium equations, as proposed by Tessler [8].

For thick plates, the in-plane displacements tend to exhibit non-linear behavior across the thickness. For this reason, a linear piece-wise model for the analysis of thick laminates is not sufficient and a higher-order kinematics assumption has to be adopted. In the framework of the zigzag theories, Di Sciuva [9] was the first to formulate a third-order zigzag plate model able to satisfy *a priori* the stress continuity conditions at layers interfaces. Later, by using some theoretical results proper of the RZT, Nemeth [10] developed a cubic zigzag model wherein the continuity conditions on transverse shear stresses are not enforced, thus providing a piecewise quadratic distribution of these stresses.

In order to include the thickness deformation in relatively thick laminates, Barut et al. [11] enriched the RZT in-plane displacements with a piecewise quadratic through-the-thickness contribution and a quadratic distribution of the transverse displacement along the laminate thickness. Moreover, a cubic polynomial pattern for the transverse normal stress is assumed independently without enforcing the traction conditions on external plate surfaces, thus providing not accurate constitutive transverse normal stress.

In this paper, a novel Mixed Cubic Zigzag Theory ($CZT^{(m)}$) is developed via the Reissner Mixed Variational Theorem. The kinematic assumption results in a piece-wise cubic through-the-thickness distribution of the in-plane displacements and the transverse displacement varying quadratically along the entire laminate thickness. The assumed transverse shear stresses profile is derived from integration of the three-dimensional equilibrium equations as performed in [6] whereas the transverse normal stress is assumed smeared cubically distributed along the laminate thickness. The model results in a nine-kinematic variables independent of the number of layers and no shear correction factors are required.

2 $CZT^{(m)}$ ASSUMPTIONS

The development of a zigzag model that does not enforce *a priori* the continuity condition on transverse shear stresses leads to a piece-wise continuous through-the-thickness distribution of the constitutive transverse shear stresses, thus violating the elasticity continuity requirements. A way to avoid the integration of the three-dimensional equilibrium equations to derive continuous transverse stresses is to develop a mixed model, via the Reissner Mixed Variational Theorem, wherein the transverse stresses are assumed independently of the displacements. The assumed stresses profile is continuous along the thickness and able to satisfy the traction conditions on the external plate surfaces.

According to [7], the stationary condition of Reissner’ functional read as:

$$\int_{\Omega}\int_{-h}^{h}\left[\delta\boldsymbol{\varepsilon}^{T}\boldsymbol{\sigma}+\delta\boldsymbol{\sigma}_{\mathbf{a}}^{T}\left(\boldsymbol{\varepsilon}_{\mathbf{t}}-\boldsymbol{\varepsilon}_{\mathbf{a}}\right)\right]dzdS=\delta W_{e} \tag{1}$$

where δ is the variational operator and W_e represent the work of the external loads, respectively. The strain vector $\boldsymbol{\varepsilon}^T=\left\{\varepsilon_{11},\varepsilon_{22},\varepsilon_{zz},\gamma_{12},\gamma_{1z},\gamma_{2z}\right\}$ collects strains computed by means of the linear strain-displacement relations, the stress vector $\boldsymbol{\sigma}^T=\left\{\sigma_{11},\sigma_{22},\sigma_{zz}^a,\tau_{12},\tau_{1z}^a,\tau_{2z}^a\right\}$ contains in-plane stresses obtained by means of the Hooke's law and the assumed transverse stresses (superscript *a*). Similarly, the vector $\boldsymbol{\sigma}_{\mathbf{a}}^T=\left\{\tau_{1z}^a,\tau_{2z}^a,\sigma_{zz}^a\right\}$ collects the assumed transverse stresses, $\boldsymbol{\varepsilon}_{\mathbf{t}}^T=\left\{\gamma_{1z},\gamma_{2z},\varepsilon_{zz}\right\}$ and $\boldsymbol{\varepsilon}_{\mathbf{a}}^T=\left\{\gamma_{1z}^a,\gamma_{2z}^a,\varepsilon_{zz}^a\right\}$ contains the transverse strains coming from the strain-displacement relations using CZT$^{(m)}$ kinematic and those coming from the Hooke's law by using the assumed stresses, respectively.

In the following sections, the assumed displacement field and the assumed transverse stress one are described.

2.1 Displacement field

Consider a laminated plate of uniform thickness *2h* with *N* perfectly bonded orthotropic layers (see Figure 1). As reference frame, a orthogonal Cartesian coordinate system ($\mathbf{x},z$) is assumed, wherein $\mathbf{x}$ denotes the order pair (x_1,x_2). The midplane of the plate, denoted with $\boldsymbol{\Omega}$, is locate on the $\mathbf{x}$-plane and the *z* coordinate ranges from *–h* to *h*. The cylindrical surface that bounds the plate, *S*, is composed by S_u and S_σ, wherein the kinematics and forces boundary conditions are applied, respectively.

The orthogonal components of the displacement vector in the *k*th layer (superscript *(k)*) are expressed as

$$\begin{aligned}U_{\alpha}^{(k)}(\mathbf{x},z)&=u_{\alpha}(\mathbf{x})+z\theta_{\alpha}(\mathbf{x})+z^{2}\chi_{\alpha}(\mathbf{x})+z^{3}\omega_{\alpha}(\mathbf{x})+\phi_{\alpha}^{(k)}(z)\psi_{\alpha}(\mathbf{x})\\U_{z}(\mathbf{x},z)&=H_{b}^{w}(z)w_{b}(\mathbf{x})+H_{t}^{w}(z)w_{t}(\mathbf{x})+H_{a}^{w}(z)\overline{w}(\mathbf{x})\end{aligned} \tag{2}$$

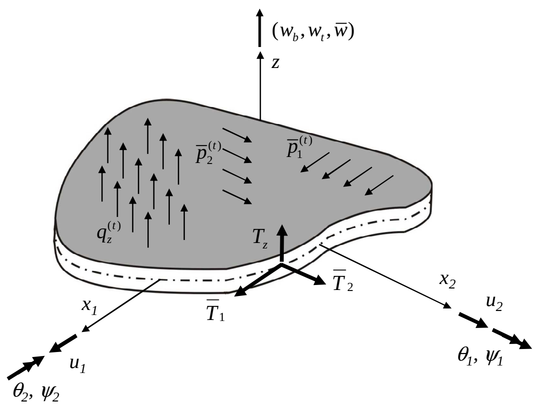

Figure 1. General plate notation.

The kinematic assumptions in Eq. (2) are an enrichment of the RZT displacement field [4]: the in-plane displacement components, $U_\alpha^{(k)}$ ($\alpha = 1,2$), are given by the superposition of RZT in plane-displacements and a quadratic and a cubic smeared contribution [12]. Thus, the u_α, θ_α, ψ_α and $\phi_\alpha^{(k)}$ ($\alpha = 1,2$) represent, respectively, the uniform in-plane displacement, the rotation along the β-axis, the zigzag amplitude and the RZT zigzag function [4]. The χ_α and ω_α are regarded as higher-order rotations, accounting for the actual deformation of the normal in a thick plate. Instead, the transverse displacement, $U_z(\mathbf{x}, z)$, is assumed to vary in a ESL-view quadratically along the thickness direction. Thus, w_b, w_t and $\bar{w}$ are the bottom, top and average transverse displacements respectively, and the $H_b^w(z)$, $H_t^w(z)$ and $H_a^w(z)$ are

$$H_b^w(z) = -\frac{1}{4} - \frac{1}{2h}z + \frac{3}{4h^2}z^2; \quad H_t^w(z) = -\frac{1}{4} + \frac{1}{2h}z + \frac{3}{4h^2}z^2; \quad H_a^w(z) = \frac{3}{2} - \frac{3}{2h^2}z^2 \tag{3}$$

2.1.1 Derivation of the zigzag function

The CZT$^{(m)}$ kinematic assumptions involve thirteen kinematic variables, independent from the number of layers. In order to reduce the number of degrees of freedom, reducing further the computational cost, a novel higher-order zigzag function is introduced wherein to the piece-wise linear contribution a smeared quadratic and cubic one is added, thus resulting in a piece-wise cubic function.

Consistent with the displacement field in Eq.(2) and according to the linear strain-displacement relations, the transverse shear strain read as

$$\gamma_{\alpha z}^{(k)} = U_{\alpha,z}^{(k)} + U_{z,\alpha} = \left(1 + \phi_{\alpha,z}^{(k)}(z)\right)\psi_\alpha + 2z\chi_\alpha + 3z^2\omega_\alpha + \eta_\alpha \tag{4}$$

where the strain measure $\eta_\alpha = \theta_\alpha + U_{z,\alpha} - \psi_\alpha$ is introduced. According to the Hooke's law for the kth lamina whose principal material directions are arbitrary with respect to the midplane reference coordinates $\mathbf{x} \in \Omega$, the transverse shear stresses are related to the transverse shear strains as

$$\tau_{\alpha z}^{(k)} = Q_{\alpha\beta}^{(k)}\gamma_{\beta z}^{(k)} = Q_{\alpha\beta}^{(k)}\left[\left(1 + \phi_{\beta,z}^{(k)}(z)\right)\psi_\beta + 2z\chi_\beta + 3z^2\omega_\beta\right] + Q_{\alpha\beta}^{(k)}\eta_\beta \tag{5}$$

where $Q_{\alpha\beta}^{(k)}$ are the shear elastic stiffness coefficients. To completely define the zigzag function, three conditions are enforced on the transverse shear stresses component (Eq.(5)) obtained vanishing η_α [12]:

1. The zero condition on the top and bottom plate surface, that is

$$\begin{cases} Q_{\alpha\alpha}^{(k)}\left[\left(1 + \phi_{\alpha,z}^{(k)}(-h)\right)\psi_\alpha - 2h\chi_\alpha + 3h^2\omega_\alpha\right] = 0 \\ Q_{\alpha\alpha}^{(k)}\left[\left(1 + \phi_{\alpha,z}^{(k)}(h)\right)\psi_\alpha + 2h\chi_\alpha + 3h^2\omega_\alpha\right] = 0 \end{cases} \tag{6}$$

Solving Eq.(6), the kinematic variables χ_α and ω_α are expressed in terms of the zigzag amplitude ψ_α, that is

$$\chi_\alpha = -\chi_\alpha^{(0)}\psi_\alpha; \quad \omega_\alpha = -\omega_\alpha^{(0)}\psi_\alpha \tag{7}$$

wherein $\chi_\alpha^{(0)}$ and $\omega_\alpha^{(0)}$ are scalar easily deducible.

2. Substituting Eqs. (7) in Eq.(5), transverse shear stress becomes

$$\tau_{\alpha z}^{(k)} = Q_{\alpha\beta}^{(k)}\left[\left(1+\phi_{\beta,z}^{(k)}(z)\right) - 2z\chi_\alpha^{(0)} - 3z^2\omega_\alpha^{(0)}\right]\psi_\beta + Q_{\alpha\beta}^{(k)}\eta_\beta \tag{8}$$

Similar to the RZT [4], the partial continuity condition at layers interface is enforced. This constrain read as

$$\begin{aligned} &Q_{\alpha\alpha}^{(k)}\left[\left(1+\phi_{\alpha,z}^{(k)}({}^{+}z^{(k)})\right) - 2\left({}^{+}z^{(k)}\right)\chi_\alpha^{(0)} - 3\left({}^{+}z^{(k)}\right)^2\omega_\alpha^{(0)}\right] = \\ &Q_{\alpha\alpha}^{(k+1)}\left[\left(1+\phi_{\alpha,z}^{(k+1)}({}^{-}z^{(k+1)})\right) - 2({}^{-}z^{(k+1)})\chi_\alpha^{(0)} - 3({}^{-}z^{(k+1)})^2\omega_\alpha^{(0)}\right] \end{aligned} \tag{9}$$

where ${}^{+}z^{(k)}$ and ${}^{-}z^{(k+1)}$ are the top and bottom interface of *k*th and *k*th+1 layers, respectively.

By introducing the constraint of Eq.(7) in the displacement field, the in-plane displacement components become

$$U_\alpha^{(k)}(\mathbf{x},z) = u_\alpha(\mathbf{x}) + z\theta_\alpha(\mathbf{x}) + \left(-z^2\chi_\alpha^{(0)} - z^3\omega_\alpha^{(0)} + \phi_\alpha^{(k)}(z)\right)\psi_\alpha(\mathbf{x}) \tag{10}$$

Then, the definition of the cubic zigzag function follows

$$\mu_\alpha^{(k)} = \left(-z^2\chi_\alpha^{(0)} - z^3\omega_\alpha^{(0)} + \phi_\alpha^{(k)}(z)\right) \tag{11}$$

wherein a quadratic and a cubic smeared contributions are added to the RZT zigzag function, $\phi_\alpha^{(k)}(z)$.

3. Finally, the last constraint is the zero-condition on the external plate surfaces for the zigzag function $\mu_\alpha^{(k)}$, that is

$$\mu_\alpha^{(1)}(-h) = \mu_\alpha^{(N)}(h) = 0 \tag{12}$$

Consistent with the displacement field in Eq. (2), constrain in Eq. (7) and the definition of the zigzag function, the CZT$^{(m)}$ kinematic assumptions are rewritten in a final form that resembles the RZT formalism only for the in-plane displacements assumption

$$\begin{aligned} U_\alpha^{(k)}(\mathbf{x},z) &= u_\alpha(\mathbf{x}) + z\theta_\alpha(\mathbf{x}) + \mu_\alpha^{(k)}(z)\psi_\alpha(\mathbf{x}) \\ U_z(\mathbf{x},z) &= H_b^w(z)w_b(\mathbf{x}) + H_t^w(z)w_t(\mathbf{x}) + H_a^w(z)\overline{w}(\mathbf{x}) \end{aligned} \tag{13}$$

The strains are computed by using the linear strain-displacement relations

$$2\varepsilon_{\alpha\beta}^{(k)} = U_{\alpha,\beta}^{(k)} + U_{\beta,\alpha}^{(k)}; \quad \varepsilon_{zz} = U_{z,z}; \quad \gamma_{\alpha z}^{(k)} = \gamma_\alpha(z) + \lambda_\alpha^{(k)}\psi_\alpha \tag{14}$$

with $\gamma_\alpha(z) \equiv U_{z,\alpha} + \theta_\alpha$ and $\lambda_\alpha^{(k)} \equiv \mu_{\alpha,z}^{(k)}$. A mixed form of the Hooke's law [13] is used, according to which, the stresses are related with the strains as

$$\sigma_{\alpha\beta}^{(k)} = \tilde{Q}_{\alpha\beta\gamma\delta}^{(k)}\varepsilon_{\gamma\delta}^{(k)} + S_{33}^{(k)}R_{\alpha\beta}^{(k)}\sigma_{zz}^{a}; \quad \tau_{\alpha z}^{(k)} = Q_{\alpha\beta}^{(k)}\gamma_{\beta z}^{(k)} \tag{15}$$

where $\tilde{Q}^{(k)}_{\alpha\beta\gamma\delta}$ and $Q^{(k)}_{\alpha\beta}$ are the transformed elastic stiffness coefficients referred to the $(\mathbf{x}, z)$ coordinate system and relative to the plane-stress condition that assumes that transverse normal stress is negligibly small in relation to the in-plane stresses. The transverse normal compliance $S^{(k)}_{33}$ and the elastic stiffness coefficients matrix regarding the transverse normal contribution, are defined as

$$S^{(k)}_{33} = \frac{1}{C^{(k)}_{33}}; \quad R^{(k)}_{\alpha\beta} = \begin{bmatrix} C_{13} & C_{36} \\ C_{36} & C_{23} \end{bmatrix}^{(k)} \tag{16}$$

wherein $C^{(k)}_{33}, C^{(k)}_{13}, C^{(k)}_{23}, C^{(k)}_{36}$ are the elastic stiffness coefficients. Finally, the transverse normal deformation, ε^{a}_{zz} , computed by means of the mixed form of the Hooke's law read as

$$\varepsilon^{a}_{zz} = -S^{(k)}_{33} R^{(k)}_{\alpha\beta} \varepsilon^{(k)}_{\gamma\delta} + S^{(k)}_{33} \sigma^{a}_{zz} \tag{17}$$

2.2 Transverse stresses

The assumed transverse stresses are continuous along the thickness and able to satisfy the traction conditions on the external plate surfaces. Two different strategies are involved in the approximation of the transverse normal stress and the transverse shear ones.

2.2.1 Transverse normal stress

The CZT$^{(m)}$ model assumes a smeared cubic through-the-thickness distribution of the transverse normal stress that resembles the distribution provided by the exact Elasticity solution. In this model, the assumed transverse normal stress is

$$\sigma^{a}_{zz} = \mathbf{P}(z)\mathbf{q}_{\mathbf{V}} + \mathbf{L}(z)\mathbf{q}_{\mathbf{z}} \tag{18}$$

Wherein the vector $\mathbf{q_z}^T = \left\{ q^{(b)}_z, q^{(t)}_z \right\}$ collects the external transverse normal pressure applied at the bottom and top plate surface, the stress vector $\mathbf{q}_{\mathbf{V}}$ collects two kinematic variables-independent coefficients, the vector $\mathbf{P}(z)$ and $\mathbf{L}(z)$ rule the shape of the assumed stress and are defined as follows

$$\mathbf{P}(z) = \left\{ h^2 \left[\left(\frac{z}{h} \right)^2 - 1 \right] \quad zh^2 \left[\left(\frac{z}{h} \right)^2 - 1 \right] \right\}; \quad \mathbf{L}(z) = \left\{ \frac{1}{2} \left(\frac{z}{h} - 1 \right) \quad \frac{1}{2} \left(\frac{z}{h} + 1 \right) \right\} \tag{19}$$

2.2.2 Transverse shear stresses

The assumed transverse shear stresses profile is derived by integration of the three-dimensional equilibrium equations under the cylindrical bending assumption, following the approach proposed by Tessler [8] and adopted in the development of the RZT$^{(m)}$ plate model [6]. According to this procedure, the assumed transverse shear stresses are expressed as

$$\boldsymbol{\tau}_{\mathbf{a}} = \mathbf{Z}_{\mathbf{f}}(z)\mathbf{f}_{\mathbf{v}}(\mathbf{x}) + \mathbf{Z}_{\mathbf{n}}(z)\mathbf{n}_{\mathbf{v}}(\mathbf{x}) + \mathbf{Z}_{\mathbf{q}}(z)\mathbf{n}_{\mathbf{q}}(\mathbf{x}) \tag{20}$$

where $\boldsymbol{\tau}_{\mathbf{a}}^{T}=\left\{\tau_{1z}^{a},\tau_{2z}^{a}\right\}$ is the vector of the assumed stresses, the matrices $\mathbf{Z}_{\mathbf{f}}$, $\mathbf{Z}_{\mathbf{n}}$ and $\mathbf{Z}_{\mathbf{q}}$ are dependent on the thickness coordinate, $\mathbf{f}_{\mathbf{v}}$ is a stress vector function of the in-plane coordinates and kinematic variable-independent; $\mathbf{n}_{\mathbf{v}}^{T}=\left\{p_{1}^{(b)},p_{2}^{(b)},p_{1}^{(t)},p_{2}^{(t)}\right\}$ is a vector containing prescribed surface tractions applied on the top *(t)* and bottom *(b)* plate surface and acting along the x_{α} -direction; $\mathbf{n}_{\mathbf{q}}^{T}=\left\{q_{z,1}^{(b)},q_{z,2}^{(b)},q_{z,1}^{(t)},q_{z,2}^{(t)}\right\}$ is a vector containing the derivatives with respect to the x_{α} -direction of the transverse pressures applied on the external plate surfaces.

3 CZT$^{(m)}$ GOVERNING EQUATIONS

The plate is subjected to a transversely distributed pressure and to tangential distributed loads acting along the in-plane coordinate axes, applied on the external surface (see Figure 1). The body forces are neglected.

In order to allow independent assumption on displacements and transverse stresses, the Reissner' functional enforces a compatibility constrain (second contribution on the left-hand side of Eq. (1)) between the transverse strains coming from the displacement field and those coming from the assumed stresses by means of the Hooke's law. Since the assumed stresses are kinematic variables-independent, the Reissner Mixed Variational Theorem can be splitted in two contributions that are solved separately, that is

$$\int_{\Omega}\int_{-h}^{h}\delta\boldsymbol{\varepsilon}^{T}\boldsymbol{\sigma}\;dzdS=\delta W_{e} \tag{21.1}$$

$$\int_{\Omega}\int_{-h}^{h}\delta\boldsymbol{\sigma}_{\mathbf{a}}^{T}\left(\boldsymbol{\varepsilon}_{\mathbf{t}}-\boldsymbol{\varepsilon}_{\mathbf{a}}\right)\;dzdS=0 \tag{21.2}$$

Solution of Eq. (21.1) leads to the governing equations and variationally consistent boundary conditions whereas the compatibility constrain (Eq. (21.2)) is used to derive an expression of vectors $\mathbf{q}_{\mathbf{V}}$ (Eq.(18)) and $\mathbf{f}_{\mathbf{V}}$ (Eq.(20)) in terms of the kinematic variables that ensure the fulfillment of the compatibility constrain.

3.1 Governing equations and variationally consistent boundary conditions

Consistent with the CZT$^{(m)}$ displacement field, the strain components are computed and their expression introduced in Eq. (21.1). Performing the integration by parts, the governing equations are obtained (all details are omitted for sake of brevity)

$$\begin{array}{lll}\delta u_{\alpha}:N_{\alpha\beta,\beta}+\overline{p}_{\alpha}=0; & \delta\theta_{\alpha}:M_{\alpha\beta,\beta}-V_{\alpha z}+\overline{m}_{\alpha}=0; & \delta\psi_{\alpha}:M^{\mu}_{\alpha\beta,\beta}-V^{\mu}_{\alpha z}=0;\\ \delta w_{b}:V^{b}_{\alpha z,\alpha}-N^{b}_{z}+q^{(b)}_{z}=0; & \delta w_{t}:V^{t}_{\alpha z,\alpha}-N^{t}_{z}+q^{(t)}_{z}=0; & \delta\overline{w}:V^{m}_{\alpha z,\alpha}-N^{m}_{z}=0\end{array} \tag{22}$$

where the resultants of the applied loads $\overline{p}_{\alpha}=p_{\alpha}^{(b)}+p_{\alpha}^{(t)}$, $\overline{m}_{\alpha}=h\left(p_{\alpha}^{(t)}-p_{\alpha}^{(t)}\right)$ are introduced. Moreover, the following forces and moments stress resultants are defined

$$\begin{aligned}&\left(N_{\alpha\beta},M_{\alpha\beta},M_{\alpha\beta}^{\mu}\right)=\int_{-h}^{h}\sigma_{\alpha\beta}^{(k)}(1,z,\mu_{\alpha}^{(k)})dz;\quad \left(V_{\alpha z},V_{\alpha z}^{\mu}\right)=\int_{-h}^{h}\tau_{\alpha z}^{a}(1,\mu_{\alpha}^{(k)})dz\\&\left(V_{\alpha z}^{b},V_{\alpha z}^{t},V_{\alpha z}^{m}\right)=\int_{-h}^{h}\tau_{\alpha z}^{a}(H_{b}^{w}(z),H_{t}^{w}(z),H_{a}^{w}(z))dz;\quad \left(N_{z}^{b},N_{z}^{t},N_{z}^{m}\right)=\int_{-h}^{h}\sigma_{zz}^{a}(H_{b,z}^{w}(z),H_{t,z}^{w}(z),H_{a,z}^{w}(z))\end{aligned}\tag{23}$$

Solving Eq. (21.1) leads also to the variationally consistent boundary conditions

$$\begin{aligned}&u_{\alpha}=\bar{u}_{\alpha}\ \wedge\ N_{\alpha\beta}n_{\beta}=\bar{N}_{\alpha n};\quad \theta_{\alpha}=\bar{\theta}_{\alpha}\ \wedge\ M_{\alpha\beta}n_{\beta}=\bar{M}_{\alpha n};\quad \psi_{\alpha}=\bar{\psi}_{\alpha}\ \wedge\ M_{\alpha\beta}^{\mu}n_{\beta}=\bar{M}_{\alpha n}^{\mu}\\&w_{b}=\bar{w}_{b}\ \wedge\ V_{\alpha z}^{b}n_{\alpha}=\bar{V}_{zn}^{b};\quad w_{t}=\bar{w}_{t}\ \wedge\ V_{\alpha z}^{t}n_{\alpha}=\bar{V}_{zn}^{t}\end{aligned}\tag{24}$$

wherein the kinematic conditions are applied on S_u and the force conditions on S_σ. Moreover, the force and moment resultants of the prescribed tractions, denoted with a bar in Eqs. (24), are defined as

$$\left(\bar{N}_{\alpha n},\bar{M}_{\alpha n},\bar{M}_{\alpha n}^{\mu},\bar{V}_{zn}^{b},\bar{V}_{zn}^{t}\right)=\int_{-h}^{h}\left(\bar{T}_{\alpha},z\bar{T}_{\alpha},\mu_{\alpha}^{(k)}\bar{T}_{\alpha},H_{b}^{w}(z)T_{z}\ ,H_{t}^{w}(z)T_{z}\ \right)\tag{25}$$

3.2 Compatibility constraint

The assumed transverse stresses, Eqs.(18,20), depend on vectors stress, $\mathbf{q_V}$ and $\mathbf{f_V}$, that are independent on each other. This allows for the possibility to satisfy the Reissner' constraint on the transverse normal stress separately from that regarding the transverse shear stresses, that is

$$\int_{\Omega}\int_{-h}^{h}\delta\sigma_{zz}^{a\,T}\left(\varepsilon_{zz}-\varepsilon_{zz}^{a}\right)dzdS=0\tag{26.1}$$

$$\int_{\Omega}\int_{-h}^{h}\delta\boldsymbol{\tau}_{\mathbf{a}}^{\ T}\left(\boldsymbol{\gamma}_{\mathbf{t}}-\boldsymbol{\gamma}_{\mathbf{a}}\right)\ dzdS=0\tag{26.2}$$

wherein $\boldsymbol{\gamma}_{\mathbf{t}}^{\ T}=\left\{\gamma_{1z},\gamma_{2z}\right\},\boldsymbol{\gamma}_{\mathbf{a}}^{\ T}=\left\{\gamma_{1z}^{a},\gamma_{2z}^{a}\right\}$. By using Eqs. (14),(17),(18) and (20), the constraints are expressed in the following form

$$\int_{\Omega}\int_{-h}^{h}\delta\mathbf{q_V}^{T}\mathbf{P}(z)^{T}\left[U_{z,z}+S_{33}^{(k)}R_{\alpha\beta}^{(k)}\varepsilon_{\gamma\delta}^{(k)}-S_{33}^{(k)}\left(\mathbf{P}(z)\mathbf{q_V}+\mathbf{L}(z)\mathbf{q_z}\right)\right]dzdS=0\tag{27.1}$$

$$\int_{\Omega}\int_{-h}^{h}\delta\mathbf{f}_{\mathrm{v}}(\mathbf{x})^{T}\mathbf{Z_f}(z)^{T}\left[\boldsymbol{\gamma}(z)+\boldsymbol{\lambda}^{(k)}\boldsymbol{\Psi}-\mathbf{D_t}\left(\mathbf{Z_f}(z)\mathbf{f}_{\mathrm{v}}(\mathbf{x})+\mathbf{Z_n}(z)\mathbf{n}_{\mathrm{v}}(\mathbf{x})+\mathbf{Z_q}(z)\mathbf{n_q}(\mathbf{x})\right)\right]\ dzdS=0\tag{27.2}$$

where $\boldsymbol{\gamma}(z)^{T}=\left\{\gamma_1(z),\gamma_2(z)\right\}$, $\boldsymbol{\Psi}^{T}=\left\{\psi_1,\psi_2\right\}$ and $\boldsymbol{\lambda}^{(k)}=diag(\mu_{1,1}^{(k)},\mu_{2,2}^{(k)})$ is a diagonal matrix containing the derivatives of the zigzag functions $\mu_{\alpha}^{(k)}$ with respect to the in-plane coordinate axes. Solving Eqs.(27) with respect to the stress vectors, $\mathbf{q_V}$ and $\mathbf{f_V}$, and substituting the results in Eq. (18) and Eq. (20), the form of the assumed stresses satisfying the compatibility constrain is obtained

$$\sigma_{zz}^{a}=\mathbf{S^0}(z)\boldsymbol{\varepsilon}_{\mathbf{p}}^{\mathbf{(0)}}+\mathbf{S^1}(z)\mathbf{k}+\mathbf{S}^{\boldsymbol{\mu}}(z)\boldsymbol{\psi}+\mathbf{S}^{\mathrm{w}}(z)\mathbf{w}+\mathbf{S^{qz}}(z)\mathbf{q_z}\tag{28.1}$$

$$\boldsymbol{\tau}_a = \mathbf{Z}_{\boldsymbol{\theta}}(z)\boldsymbol{\theta} + \mathbf{Z}_{\psi}(z)\boldsymbol{\Psi} + \mathbf{Z}_{\mathbf{eq}}(z)\mathbf{n}_{\mathbf{q}} + \mathbf{Z}_{\mathbf{ep}}(z)\mathbf{n}_{\mathbf{v}} + \mathbf{Z}_{\mathbf{w}}(z)\mathbf{w}_{\mathbf{d}} \tag{28.2}$$

where $\mathbf{S}^{\mathbf{0}}(z), \mathbf{S}^{\mathbf{1}}(z), \mathbf{S}^{\boldsymbol{\mu}}(z), \mathbf{S}^{\mathbf{w}}(z), \mathbf{S}^{\mathbf{qz}}(z)$ and $\mathbf{Z}_{\boldsymbol{\theta}}(z), \mathbf{Z}_{\psi}(z), \mathbf{Z}_{\mathbf{eq}}(z), \mathbf{Z}_{\mathbf{ep}}(z), \mathbf{Z}_{\mathbf{w}}(z)$ are the assumed transverse stresses shape functions, continuous along the thickness direction and able to satisfy the traction conditions on the top and bottom plate surface. Moreover, $\boldsymbol{\varepsilon}_{\mathbf{p}}^{(0)T} = \{u_{1,1} \quad u_{1,2} \quad u_{2,1} \quad u_{2,2}\}$, $\mathbf{k}^T = \{\theta_{1,1} \quad \theta_{1,2} \quad \theta_{2,1} \quad \theta_{2,2}\}$, $\boldsymbol{\psi}^T = \{\psi_{1,1} \quad \psi_{1,2} \quad \psi_{2,1} \quad \psi_{2,2}\}$, $\mathbf{w}^T = \{w_b \quad w_t \quad \bar{w}\}$, $\boldsymbol{\theta}^T = \{\theta_1 \quad \theta_2\}$ and $\mathbf{w_d}^T = \{w_{b,1} \quad w_{t,1} \quad \bar{w}_{,1} \quad w_{b,2} \quad w_{t,2} \quad \bar{w}_{,2}\}$.
It is worth to note that, the CZT$^{(m)}$ model, even be a mixed one, retains as variables the only kinematic ones.

3.3 CZT$^{(m)}$ constitutive equations

Introducing Eqs. (15) in the definition of the forces and moments stress resultants, Eqs. (23), and making use of the linear strain-displacement relations, Eqs. (14), the CZT$^{(m)}$ plate constitutive equations are derived. In matrix form, they appear as

$$\begin{Bmatrix} \mathbf{N} \\ \mathbf{M} \\ \mathbf{M}^{\boldsymbol{\mu}} \\ \mathbf{N_z} \end{Bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B_n} & \mathbf{C_n} & \mathbf{A_w} & \mathbf{A_{qz}} \\ \mathbf{B_m} & \mathbf{D} & \mathbf{E_m} & \mathbf{B_w} & \mathbf{B_{qz}} \\ \mathbf{C_{mn}} & \mathbf{F_{mn}} & \mathbf{G_{mn}} & \mathbf{C_w} & \mathbf{C_{qz}} \\ \mathbf{A_z} & \mathbf{B_z} & \mathbf{C_z} & \mathbf{D_z} & \mathbf{E_z} \end{bmatrix} \mathbf{e_b}; \quad \begin{Bmatrix} \mathbf{V} \\ \mathbf{V}^{\boldsymbol{\mu}} \\ \mathbf{V_z} \end{Bmatrix} = \begin{bmatrix} \mathbf{A_T} & \mathbf{B_T} & \mathbf{C_T} & \mathbf{D_T} & \mathbf{E_T} \\ \mathbf{F_T} & \mathbf{G_T} & \mathbf{H_T} & \mathbf{L_T} & \mathbf{N_T} \\ \mathbf{M_T} & \mathbf{U_T} & \mathbf{P_T} & \mathbf{W_T} & \mathbf{J_T} \end{bmatrix} \mathbf{e_s} \tag{28}$$

where $\mathbf{e_b} = \{\boldsymbol{\varepsilon}_{\mathbf{p}}^{(0)} \quad \mathbf{k} \quad \boldsymbol{\psi} \quad \mathbf{w} \quad \mathbf{q_z}\}^T$, $\mathbf{e_s} = \{\boldsymbol{\theta} \quad \boldsymbol{\Psi} \quad \mathbf{w_d} \quad \mathbf{n_z} \quad \mathbf{n_v}\}^T$, the forces and moments stress resultants are organized as $\mathbf{N}^T = \{N_{11} \quad N_{12} \quad N_{21} \quad N_{22}\}$, $\mathbf{M}^T = \{M_{11} \quad M_{12} \quad M_{21} \quad M_{22}\}$, $\mathbf{M}^{\boldsymbol{\mu}T} = \{M_{11}^{\mu} \quad M_{12}^{\mu} \quad M_{21}^{\mu} \quad M_{22}^{\mu}\}$, $\mathbf{N_z}^T = \{N_z^b \quad N_z^t \quad N_z^m\}$, $\mathbf{V}^T = \{V_{1z} \quad V_{2z}\}$, $\mathbf{V}^{\boldsymbol{\mu}T} = \{V_{1z}^{\mu} \quad V_{2z}^{\mu}\}$ and $\mathbf{V_z}^T = \{V_{1z}^b \quad V_{1z}^t \quad V_{1z}^m \quad V_{2z}^b \quad V_{2z}^t \quad V_{2z}^m\}$. The definitions of the stiffness matrices come directly from the substitution of Eqs. (15), (18) and (20) in Eqs. (23).

4 NUMERICAL RESULT

In order to assess the accuracy of the CZT$^{(m)}$ model, a simply supported on all edges rectangular plate, subjected to bi-sinusoidal transverse pressure load applied on the top plate surface, is considered. Results provided by CZT$^{(m)}$ are compared with the exact 3D-Elasticity solution as derived by Pagano [14].

The rectangular plate (*b/a*=2, *a/2h*=5) is composed by equal four layers, with orientation (0°/90°/90°/0°), while the material mechanical properties are E_1=5.9 GPa, E_2=E_3= 10 GPa; G_{12}= 5.9 GPa, G_{13}=0.2 GPa, G_{23}=0.7 GPa; $\nu_{12} = \nu_{13} = \nu_{23}$=0.25. For this type of load and boundary conditions, the exact CZT$^{(m)}$ solution exists and is derived by approximating the kinematic variables with trigonometric functions in the x_1 and x_2 direction in order to satisfy the boundary conditions.

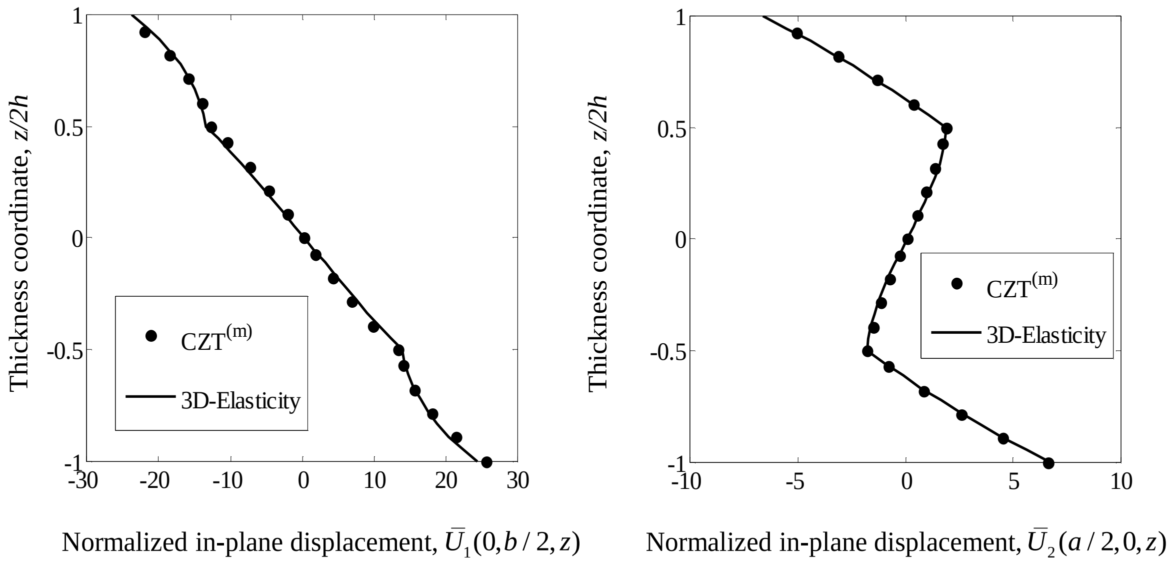

Figure 2. Through-the-thickness distribution of normalized in plane displacements, $\bar{U}_\alpha = \left(10^4 D_{11}/q_0 a^4\right) U_\alpha$.

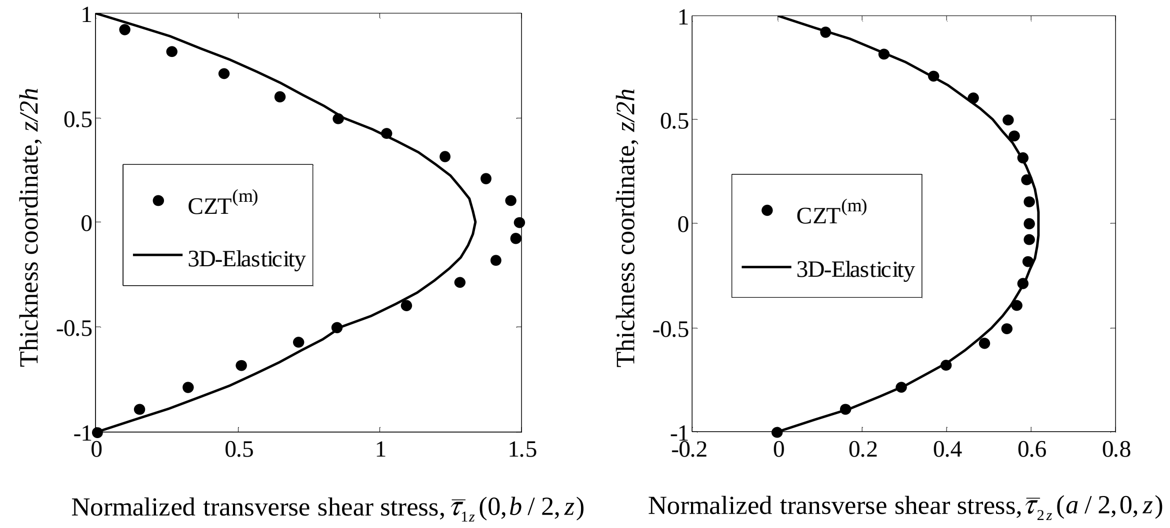

Figure 3. Through-the-thickness distribution of normalized transverse shear stresses, $\bar{\tau}_{\alpha z} = \left(2h/q_0 a^2\right) \tau_{\alpha z}$.

Figure 2 demonstrates great accuracy of the CZT$^{(m)}$ model in predicting the through-the-thickness distribution of in-plane displacements; as a consequence, the distributions of in-plane normal stresses (here omitted for brevity) provided by the proposed model fit very well with the reference solution.

Moreover, the constitutive normalized transverse shear stresses result accurate if compared with the Elasticity solution with a slight overestimation of the maximum value for the transverse shear stress τ_{1z} (see Figure 3).

The prediction of the transverse normal behavior, in terms of displacement (Figure 4a) and stress (Figure 4b) is higly accurate if compared with the reference solution, since the model captures the actual through-the-thickness distribution of transverse displacement (with an error less than 1%) and provides a distribution of normal stress that matches the Elasticity solution for every *z* location.

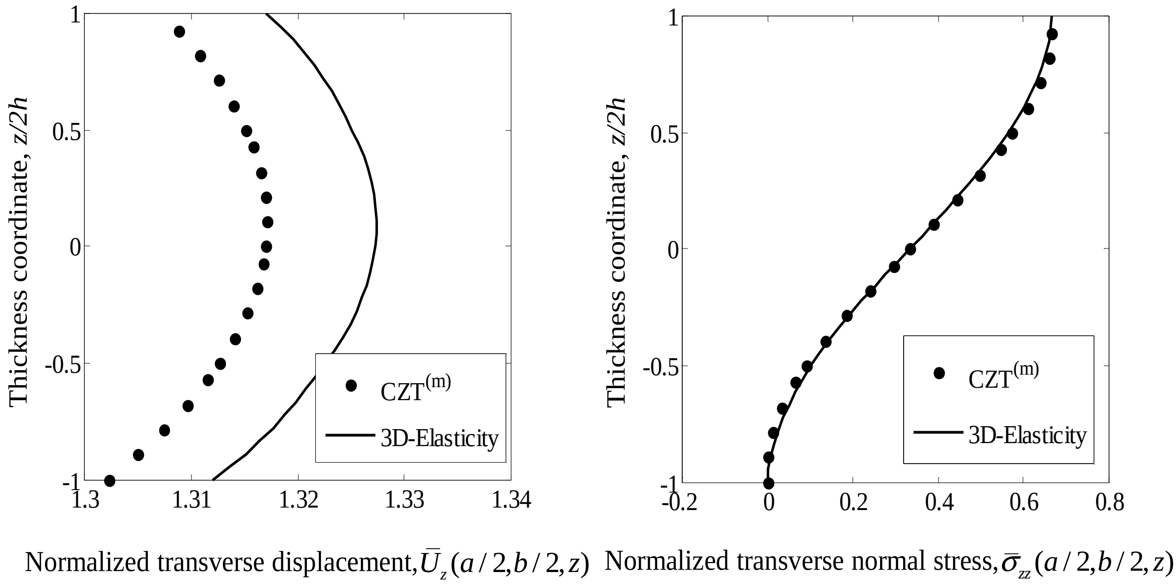

Figure 4. **(a)** Through-the-thickness distribution of normalized transverse displacement $\bar{U}_z = \left(10^2 D_{11} / q_0 a^4\right) U_z$

(b) Through-the-thickness distribution of normalized transverse normal stress $\bar{\sigma}_{zz} = \left(2h / q_0 a^2\right) \sigma_{zz}$.

5 CONCLUSIONS

In this paper, a novel Mixed Cubic Zigzag model, CZT$^{(m)}$, is presented. The development of the model is based on the Reissner Mixed Variational Theorem, which allows independent assumption for displacements and transverse stresses. The assumed displacement field results in a enrichment of the First-Order Shear Deformation Theory: to the in-plane displacements a piece-wise cubic contribution is added, whereas the transverse displacement is assumed smeared quadratic along the thickness direction. The assumed transverse shear stresses profile is derived with the aid of the three-dimensional equilibrium equations and the transverse normal stress is postulated smeared cubic. Each transverse stress profile is continuous along the thickness direction and able to satisfy the traction conditions on the top and bottom plate surface. Finally, the CZT$^{(m)}$ model results in a constant number of kinematic variables, nine, irrespective of the number of layers.

Numerical result provided in this paper shows high level of accuracy of the proposed model in the analysis of thick laminates and sandwiches if compared with the exact Elasticity solution. Due to the low computational cost and its formulation, the proposed model is suitable for an efficient finite element implementation allowing accurate large scale analysis of geometrically complex and thick multilayered composite and sandwich structures.

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