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HOMOGENIZATION OF MATERIAL PROPERTIES OF THE FGM BEAM AND SHELL FINITE ELEMENTS

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Abstract. In the contribution, the homogenization techniques of spatial varying (continuously or discontinuously) material properties for the Functionally Graded Material (FGM) beams and shells are presented. The expressions are proposed for derivation of the effective elastic, thermal and electrical material properties by the extended mixture rules and laminate theory, and by the direct integration method. The results of numerical experiments are evaluated and disscussed.

1 INTRODUCTION

Important classes of structural components, where a functionally graded material (FGM) is used, are beams and shells. FGM beams and shells play an important role not only in structural applications, but there are many applications in design of thermal-elastic, electric-thermal or electric-thermal-structural systems (e.g. MEMS like sensors and actuators, and other mechanical and mechatronic systems). In all these applications, using new materials like FGM can greatly improve the efficiency of a system. FGM is built as a mixture of two or more constituents which have almost the same geometry and dimensions. From a macroscopic point of view, the FGM is isotropic in each material point but the material properties can vary continuously or discontinuously in one, two or three directions. The variation of macroscopic material properties can be caused by varying the volume fraction of the constituents or with varying of the constituents material properties (e.g. by a non-homogeneous temperature field).

One important goal of mechanics of heterogeneous materials is to derive their effective properties from the knowledge of the constitutive laws and complex micro-structural behavior of their components. Microscopic modeling expresses the relation between the characteristics of the components of the composite and the average (effective) properties of composites. In the case of FGM it is the relation between the characteristics of the components and the effective properties of FGM.

The methods based on the homogenization theory (e.g. the mixture rules [1], [2]; self-consistent methods [3]) have been designed and successfully applied to determine the effective material properties of heterogeneous materials from the corresponding material behavior of the constituents (and of the interfaces between them) and from the geometrical arrangement of the phases. In this context, the microstructure of the material under consideration is basically taken into account by representative volume element (RVE).

Mixture rules are one of the methods for micromechanical modeling of heterogeneous materials. Improved mixture rules [4] are based on the assumption that the constituents volume fractions (formally denoted as fibres – *f* and matrix – *m*) continuously vary as the polynomial functions: $v_f(x,y,z)$ and $v_m(x.y,z)$. The condition $v_f(x,y,z)+v_m(x,y,z)=1$ must be fulfilled. Appropriated material property distribution in the real FGM is then

$$p(x,y,z)=v_f(x,y,z)p_f(x,y,z)+v_m(x,y,z)p_m(x,y,z) \tag{1}$$

Here, $p_f(x,y,z)$ and $p_m(x,y,z)$ are the spatial variations of material properties of the FGM – constituents. The assumption of the polynomial variation of the constituent's volume fractions and material properties enables an easier establishing of the main appropriated field equations and allows the modeling of many common realizable variations. An exponential law for variation of the volume fractions is also very often presented, e.g. [5], [6], [7] and many others.

In the literature and in the practical applications mostly the one directional variation of the FGM properties is presented. By the FGM beams and shells the transversal variation (continuously or discontinuously, symmetrically or asymmetrically) has been mainly considered. The homogenization of such material properties is relative simple. If the material properties varies only in longitudinal direction of the beams, the homogenization is not needed because there were new beam finite elements established, which consider such variations by very accurate and effective way [8], [9].

In the contribution, the selected homogenization techniques of spatial varying (continuously or discontinuously, symmetrically or asymmetrically) material properties for the FGM beams and shells are presented. The expressions are proposed for derivation of the effective elastic, thermal and electric material properties by the extended mixture rules and laminate theory, and by the direct integration method.

2 HOMOGENIZATION OF THE FGM BEAM

In the 2D beam finite element application the transversal and longitudinal variations of material properties are considered. The elasticity modules, the Poisson's ratio, the thermal expansion coefficient, the thermal and electrical conductivity, and the mass density are homogenized by extended mixture rules and laminate theory, and by the direct integration method.

2.1 Homogenization for the 2D beam applications by the multilayer method and laminate theory

Let us consider a two nodal straight beam element with predominantly rectangular cross-sectional area A and quadratic moment of inertia I (Figure 1). The following approach can be used also for other cross-sectional area types. The composite material of this beam arises from mixing two components (formally denoted as matrix – m and fibres – f) that are approximately of the same geometrical form and dimensions (for example made by powder metallurgy or plasma spraying).

Both the fibres volume fraction $v_f(x,y)$ and matrix volume fraction $v_m(x,y)$ are chosen as a polynomial function of x, and with continuous and symmetrical variation through its height h with respect to the neutral plane of the beam element. The volume fractions and material properties of the FGM constituents are assumed to be constant through the cross-section depth b. At each point of the beam it holds: $v_f(x,y)+v_m(x,y)=1$. The values of the volume fractions at the nodal points are denoted by indices i and j.

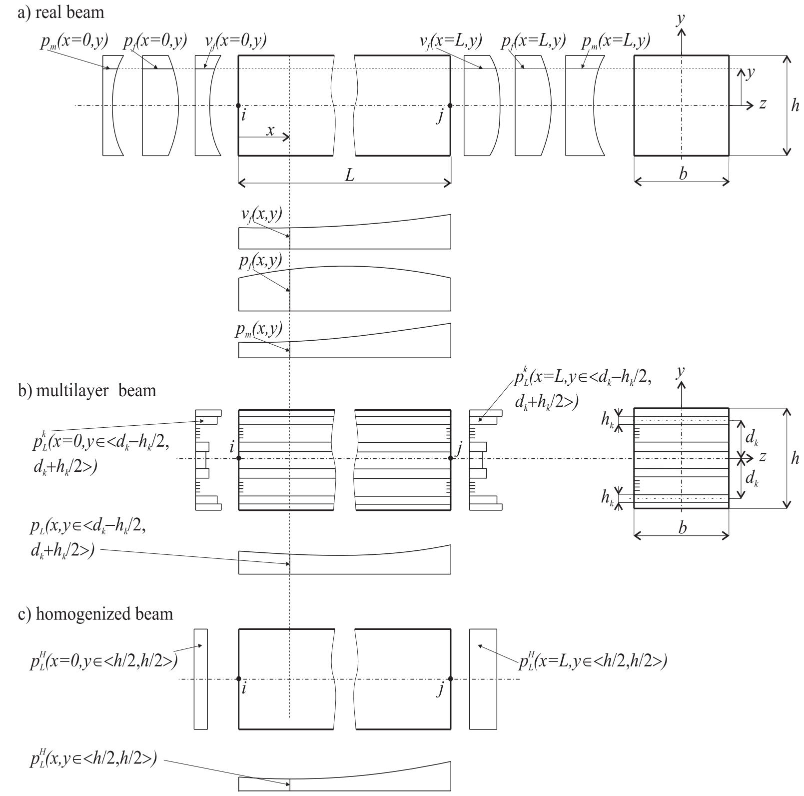

Figure 1: FGM beam with spatial variation of material properties

Material properties of the constituents (fibres – $p_f(x,y)$ and matrix – $p_m(x,y)$) can vary analogically (depending on inhomogeneous temperature field for example) as it was stated by variation of the volume fractions. A chosen FGM property in real beam $p(x,y)$ can be expressed as a function of the spatial varying volume fractions and material properties of FGM constituents

$$p(x,y) = v_f(x,y)\, p_f(x,y) + v_m(x,y)\, p_m(x,y) \tag{2}$$

For example, the planar varying elasticity modulus is [9], [10]:

$$E(x,y) = v_f(x,y) E_f(x,y) + v_f(x,y) E_m(x,y) \tag{3}$$

There $E_f(x,y)$ is the varying elasticity modulus of the fibres, and $E_m(x,y)$ is the varying elasticity modulus of the matrix. By the same way the Poisson's ratio $\nu(x,y)$ can be calculated. Then, the shear modulus reads

$$G(x,y) = \frac{E(x,y)}{2(1+\nu(x,y))} \tag{4}$$

If the constituents Poisson's ratio are approximately of the same value and the constituents volume fraction variation is not much strong, then the FGM shear modulus can by calculated using a simplification

$$G(x,y) = \frac{E(x,y)}{\xi} \tag{5}$$

where ξ is an average value of the function $\xi(x,y) = 2(1+\nu(x,y))$

$$\xi = \frac{1}{L}\int_0^L \left(\frac{1}{h} \int_{-h/2}^{h/2} \xi(x,y) dy \right) dx \tag{6}$$

Following, the electrical conductivity $\gamma(x,y)$, thermal conductivity $\lambda(x,y)$, and the thermal expansion coefficient $\alpha_T(x,y)$ can be calculated by (2), as well.

Homogenization of the material properties (the reference volume is the volume of the whole beam) will be done in two steps. In the first step, the real beam (Figure 1a) will be transformed into a multilayer beam (Figure 1b). Material properties of the layers will be calculated with the extended mixture rules [8]. Each layer will have constant volume fractions and material properties of the constituents through the beam height. They are calculated as an average value from their values at the boundaries of the respective layer. Polynomial variation of these parameters will appear in the longitudinal direction. Sufficient accuracy of the proposed substitution of the continuous lateral variation of material properties by the layer-wise constant lateral distribution of material properties will be reached when the division to layers is fine enough. In the second step, the effective longitudinal material properties of the homogenized beam will be derived using the laminate theory. These homogenized material properties are constant through the beam height but they vary continuously along the longitudinal beam axis. Accordingly, the beam finite element equation will be established for the homogenized beam (Figure 1c) in order to calculate the primary effective beam unknowns (the displacements, temperatures, electric potential, eigenfrequency, etc…).

One thin layer of the composite or FGM is depicted in Figure 2. Constant rectangular cross-sectional area of the layer has been assumed. The layers length is *L*. Longitudinal variation of the constituent volume fractions and longitudinal variation of the constituent material properties have been considered. These parameters will be considered constant through the layer height and width. The fibres (constituent 1) volume fraction $v_f(x)$ is described by polynomial function of *x*:

$$v_f(x) = 1 - v_m(x) = v_{fi}\eta_{vf}(x) = v_{fi}\left(1 + \sum_k \eta_{vfk} x^k\right) \tag{7}$$

The matrix (constituent 2) volume fraction $v_m(x)$ is then

$$v_m(x) = 1 - v_f(x) = v_{mi}\eta_{vm}(x) = v_{mi}\left(1 + \sum_k \eta_{vmk} x^k\right) \tag{8}$$

where v_f and v_m are the fibre and matrix volume fractions at node *i*, respectively. $\eta_{vf}(x)$ and $\eta_{vm}(x)$ are the polynomials of fibre and matrix volume fractions variation, respectively. Constants η_{vfk} and η_{vmk}, (*k = 1, r*), and the order *r* of these polynomials depend on the fibres and matrix volume fractions variation.

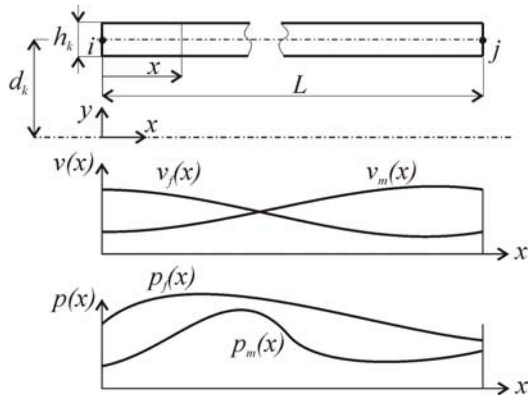

Figure 2: One thin layer of the FGM

Also the fibre material property $p_f(x)$ and the matrix material property $p_m(x)$ are chosen as a polynomial function of x:

$$p_f(x) = p_{fi}\eta_{pf}(x) = p_{fi}\left(1 + \sum_k \eta_{pfk} x^k\right) \tag{9}$$

$$p_m(x) = p_{mi}\eta_{pm}(x) = p_{mi}\left(1 + \sum_k \eta_{pmk} x^k\right) \tag{10}$$

where p_{fi} and p_{mi} are the fibre and matrix material properties at node *i*, respectively. $\eta_{pf}(x)$ is the polynomial of fibre material property variation. Its constants η_{pfk}, where $k \in \langle 1, t\rangle$, and the order *t* of this polynomial depend on the fibres material property variation. $\eta_{pm}(x)$ is the polynomial of matrix material property variation. The constants η_{pmk}, where *k = 1,…, s*, and the order *s* of this polynomial depend on the matrix material property variation.
Then the effective material property of the composite one-layer beam is given by

$$p_L(x) = v_f(x)\,p_f(x) + v_m(x)\,p_m(x) \tag{11}$$

or by polynomial form:

$$p_L(x) = p_{Li}\eta_{p_L}(x) \tag{12}$$

Here, p_{Li} is the effective longitudinal material property at node *i*, and the expression

$$\eta_{pL}(x) = \frac{p_L(x)}{p_{Li}} \tag{13}$$

is the polynomial of effective longitudinal material property variation. The notations: $p \equiv E$ – for the elasticity modulus , *G* – for the shear modulus, λ – for the thermal conductivity, γ – for the electrical conductivity, α_T – for the thermal expansion coefficient and ρ – for the mass density, respectively. Using equation (11) we obtain effective Young's modulus $E_L(x)$, effective Poisson's ratio $\nu_L(x)$, effective electrical conductivity $\gamma_L(x)$, effective thermal conductivity $\lambda_L(x)$, effective mass density $\rho_L(x)$, and effective thermal expansion coefficient $\alpha_{TL}(x)$. The more correct expression for the effective thermal expansion coefficient is [4], [8]:

$$\alpha_{TL}(x) = \frac{v_f(x)\alpha_{Tf}(x)E_f(x) + v_m(x)\alpha_{Tm}(x)E_m(x)}{v_f(x)E_f(x) + v_m(x)E_m(x)} \tag{14}$$

Expressions (6) to (8) can be used as for the effective material properties calculation of the single-layer FGM beams as for the *k*th layer of the multilayer beam.

Let us replace the initial beam (Figure 1a) by a multilayer FGM beam (Figure 1b). Lamination is symmetric according to the geometry of the layers and material properties. This symmetry allows the usage of the elementary theory of homogeneous isotropic beams for all solutions, however, material properties have to be replaced by their effective values [1]. From the mechanical coupling point of view, axial loading is not coupled with transversal loading.
Individual layers are built of FGM composite with longitudinal variation of the volume fractions and material properties of the constituents, as described above.

Homogenization of material properties of the multilayer beam will be done using the theory of laminates [1], [9], [10]. By this way we get one layer beam with a longitudinal variation of homogenized longitudinal material properties. Main dimensions of the beam - such as the beam length *L*, height *h* and width *b* – remain conserved.

For the effective common material property of the *k*th layer, according to (11), we can write:

$$p_L^k(x) = p_{Li}^k \eta_{p_L^k}(x) \tag{15}$$

Index *k* represents the layer number $(k \in \langle 1, n \rangle)$ in the upper and lower symmetrical part of the beam – see Figure 1. The number of layers of the symmetrical part is *n*. If the cross-sectional area of the *k*th layer is A_k , then the cross-sectional area ratio of the *k*th layer is defined as

$$r_{Ak} = \frac{2A_k}{A} = \frac{A_k}{\sum_{i=1}^{n} A_i} \tag{16}$$

where $2\sum_{i=1}^{n} A_i$ is the total cross-sectional area. Similarly to expression (16) we can write the moment area of inertia ratio of the *k*th layer as

$$r_{I_{zk}} = \frac{2I_{zk}}{I_z} = \frac{I_{zk}}{\sum_{i=1}^{n} I_{iz}} \tag{17}$$

where I_{zk} is the moment area of inertia of the *k*th layer and $I_z = 2\sum_{i=1}^{n} I_{zi}$ is the total area moment of inertia of cross-section. Because the Young's modulus multiplied with the cross-sectional area defines the axial stiffness and multiplied with the area moment of inertia defines the bending stiffness, we have to distinguish homogenized effective Young's modulus for axial loading $E_L^{NH}(x)$ and homogenized effective longitudinal Young's modulus for bending $E_L^{MH}(x)$. Then, the homogenized effective longitudinal common material properties $p_L^H(x)$ of the homogenized beam element can be expressed as

$$p_L^H(x) = \sum_{k=1}^{n} r_{jk}\, p_L^k(x) \tag{18}$$

where $j = A$ for the following effective longitudinal material properties: Young's modulus for axial loading $E_L^{NH}(x)$, shear modulus $G_L^H(x)$, thermal conductivity $\lambda_L^H(x)$, electrical conductivity $\gamma_L^H(x)$, mass density $\rho_L^H(x)$ and $j = I_z$ for Young's modulus for bending $E_L^{MH}(x)$. The homogenized effective longitudinal thermal expansion coefficient $\alpha_{TL}^H(x)$ of the homogenized element has to be calculated according expression [8]:

$$\alpha_{TL}^H(x) = \frac{\sum_{k-1}^{n} \alpha_{TL}^k(x) E_L^k(x) A^k}{\sum_{k-1}^{n} E_L^k(x) A^k} = \frac{1}{E_L^{NH}(x)} \sum_{k=1}^{n} r_{Ak}\, \alpha_{TL}^k(x) E_L^k(x) \tag{19}$$

For example, according the equation (18) the effective Young's modulus of the *k*th layer is

$$E_L^k(x) = E_{Li}^k \eta_{E_L^k}(x) \tag{20}$$

If the cross-sectional area of the *k*th layer is A_k, then the volume fraction of the pair of these symmetrical areas is $v_k = 2A_k / A$.

Then, the effective longitudinal elasticity modulus for axial loading of the homogenized beam can be derived using the expression

$$E_L^{NH}(x) = \sum_{k=1}^{n} v^k E_L^k(x) = E_{Li}^{NH} \eta_{E_L^{NH}}(x) \tag{21}$$

where $E_{Li}^{NH}(x) = \sum_{k=1}^{n} E_{Li}^k v^k$ is the effective longitudinal elasticity modulus for the axial loading of the homogenized beam at node *i*, and $\eta_{E_L^{NH}}(x) = E_L^{NH}(x) / E_{Li}^{NH}$ is the polynomial of its variation. This effective longitudinal elasticity modulus has to be used for the calculation of axial stiffness of the FGM beam. According to the notation in Figure 1 the effective longitudinal elasticity modulus for flexural loading of the homogenized beam of rectangular cross-section has been derived [4], [8]:

$$E_L^{MH}(x) = \frac{12}{h^3}\sum_{k=1}^{n}\left(\frac{h_k^3}{6} + 2d_k^2 h_k\right)E_L^k(x) = E_{Li}^{MH}\eta_{E_L^M}(x) \tag{22}$$

where E_{Li}^{MH} is the value of the effective longitudinal elasticity modulus in the flexural loading of the homogenized beam at node *i*, and $\eta_{E_L^{MH}} = E_L^{MH}(x)/E_{Li}^{MH}$ is the polynomial of its longitudinal variation. This effective longitudinal elasticity modulus has to be used for the calculation of the flexural stiffness of the FGM beam.

In a similar way, the effective material properties can be derived for other types of cross-sectional areas. All the homogenized effective longitudinal properties are denoted by superscript H in this chapter. As assumed, their variation along the homogenized beam is polynomial except the thermal expansion coefficient. This one has to be transformed to a polynomial by Taylor series. The multilayer method can be very effective used as by homogenization of the multilayer beam as by the continuous transversal variation of material properties. Sufficient accuracy of the substitution of continuous transversal variations of material properties by a layer-wise constant lateral distribution of material properties will be reached when division to layers is fine enough. The constant value of the material property in the assumed layer at position x will be calculated as a mean value from its values at the top and the bottom of this layer. The same method will be used also by the calculation of the components volume fractions in the competent layer.

2.2 Homogenization for the 2D beam applications by the direct integration method

By the direct integration method the transformation of FGM beam with continuously planar variation of material properties (Figure 1a) to the one layer beam with longitudinal variation of the effective material properties (Figure 1c) will be made in a single step [8], [9], [10]. From the assumption that the respective property (e.g. stiffness) of the real beam must be equal to the analogical property of the homogenized beam, the homogenized longitudinal elasticity modules for: tension – compression $E_L^{NH}(x)$, bending $E_L^{MH}(x)$, shear $G_L^H(x)$, the homogenized mass density $\rho_L^H(x)$, the electrical conductivity $\gamma_L^H(x)$, the thermal conductivity $\lambda_L^H(x)$, and the thermal expansion coeffiecient $\alpha_{TL}^H(x)$ can be calculated, respectively:

$$E_L^{NH}(x) = \frac{b\int_{-h/2}^{h/2} E(x,y)dy}{A}, \quad E_L^{MH}(x) = \frac{b\int_{-h/2}^{h/2} E(x,y)y^2 dy}{I}, \quad G_L^H(x) = \frac{b\int_{-h/2}^{h/2} G(x,y)dy}{A} \tag{23}$$

$$\rho_L^H(x) = \frac{b\int_{-h/2}^{h/2} \rho(x,y)dy}{A}, \quad \gamma_L^H(x) = \frac{\int_{-h/2}^{h/2} b\gamma(x,y)dy}{A}, \quad \lambda_L^H(x,y) = \frac{\int_{-h/2}^{h/2} b\lambda(x,y)dy}{A} \tag{24}$$

$$\alpha_{TL}^H(x) = \frac{\int_{-h/2}^{h/2} b\alpha_T(x,y)E(x,y)dy}{E_L^{NH}(x)A} \tag{25}$$

Similarly to (4), (5) and (6), the homogenized longitudinal shear modulus can be calculated by the simpler way:

$$G_L^H(x) = \frac{E_L^{NH}(x)}{\xi} \tag{26}$$

There $A = bh$ is the cross-sectional area and $I = bh^3/12$ is the moment area of inertia.
The homogenized material properties can be used for establishing of the finite element matrices of the FGM beam for single value - or muttiphysical analysis.

2.3 Homogenization for the 2D beam applications for transversal non-symmetricaly varying material properties

Consider a plane beam with a non-symmetric cross-section A, height h and width $w(z)$ with x denoting the axial direction and $z\,(-h/2 \le z \le h/2)$ referring to the transverse direction (see Figure 3).

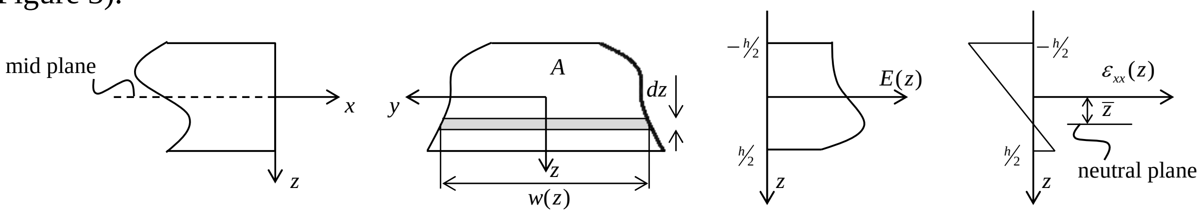

Figure 3:Homogenization of non-symmetric FGM beams

The Young's modulus E and the Poisson's ratio ν is arbitrarily distributed with respect to the geometrical mid surface $z = 0$ and membrane and bending properties are coupled. In order to avoid the derivation of coupling matrices, a neutral plane $z = \overline{z}$ is evaluated where membrane and bending properties decouple. This is done by assuming a normal strain distribution ε_{xx} caused by bending moments M_y, i.e.

$$\varepsilon_{xx}(z) = \kappa(z - \overline{z}), \tag{27}$$

with κ denoting a curvature. The normal force N_x vanishes in absence of strain offsets of the neutral plane

$$N_x = \int_A \sigma_{xx} dA = \int_{-h/2}^{h/2} E(z)\varepsilon_{xx}(z)w(z)dz = \kappa \int_{-h/2}^{h/2} E(z)(z - \overline{z})w(z)dz = 0, \tag{28}$$

leading to

$$\overline{z} = \frac{1}{\int_{-h/2}^{h/2} E(z)w(z)dz} \int_{-h/2}^{h/2} E(z)zw(z)dz \tag{29}$$

and the coordinate transformation $z' = z - \overline{z}$ can be introduced. With respect to this new transverse direction $z'\ (-h/2 - \overline{z} \le z' \le h/2 - \overline{z})$ an arbitrary strain distribution reads

$$\varepsilon_{xx}(z) = \kappa z' + \varepsilon_0, \tag{30}$$

with ε_0 denoting an axial strain of the neutral plane. The normal forces and bending moments of the strain distribution (30) read

$$N_x = \int_{-h/2-\bar{z}}^{h/2-\bar{z}} E(z')\varepsilon_0 w(z')dz' = E_m A\varepsilon_0 \; ; \; M_y = \int_{-h/2-\bar{z}}^{h/2-\bar{z}} E(z')(z')^2 \kappa w(z')dz' = E_b I\kappa \,, \tag{31}$$

defining the homogenized axial stiffness $E_m A$ and the homogenized bending stiffness $E_b I$. The shear correction factor α_s is derived from a shear strain energy balance

$$\int_A \tau_{xz}\gamma_{xz}dA = \int_{\alpha_s A} \bar{\tau}_{xz}\bar{\gamma}_{xz}dA \,, \tag{32}$$

with τ_{xz} and γ_{xz} denoting the analytically correct transverse shear stresses and transverse shear strains, respectively, which are related by

$$\tau_{xz} = \frac{E(z')}{2(1+\nu(z'))}\gamma_{xz} \,. \tag{33}$$

The constant mean values of transverse shear stresses and strains are indicated with an overbar in (32), while the following relations hold:

$$\bar{\tau}_{xz} = \bar{G}\bar{\gamma}_{xz} \quad ; \quad \bar{\tau}_{xz} = \frac{Q_z}{\alpha_s A} \,. \tag{34}$$

In (34) $\bar{G}$ refers to a constant shear modulus, i.e.

$$\bar{G} = \frac{E_b A}{\int_{-h/2-\bar{z}}^{h/2-\bar{z}} (2+2\nu(z'))w(z')dz'} \,, \tag{35}$$

And Q_z denotes the shear force. Inserting (33)-(35) in (32) leads to

$$\int_{-h/2-\bar{z}}^{h/2-\bar{z}} \frac{\tau_{xz}^2(z')2(1+\nu(z'))w(z')}{E(z')}dz' = \frac{\bar{\tau}_{xz}^2}{\bar{G}}\alpha_s A = \frac{Q_z^2}{\bar{G}\alpha_s A} \,, \tag{36}$$

and defines the shear correction factor α_s. The analytically correct shear stress distribution is found from a force equilibrium at an infinitesimal beam portion dx (see Figure. 4),

$$-\int_{z'}^{h/2-\bar{z}} \sigma_{xx}(x,\zeta)w(\zeta)d\zeta + \int_{z'}^{h/2-\bar{z}} \sigma_{xx}(x+dx,\zeta)w(\zeta)d\zeta - \tau_{xz}(z')w(z')dx = 0 \,, \tag{37}$$

which is simplified using

$$\begin{aligned} &\sigma_{xx}(x+dx,\zeta) = \sigma_{xx}(x,\zeta) + \frac{\partial\sigma_{xx}}{\partial x}dx, \\ &\sigma_{xx}(\zeta) = \mathrm{E}(\zeta)\kappa\zeta = \mathrm{E}(\zeta)\frac{M_y}{E_b I}\zeta \;\rightarrow\; \frac{\partial\sigma_{xx}}{\partial x} = \mathrm{E}(\zeta)\frac{Q_z}{E_b I}\zeta. \end{aligned} \tag{38}$$

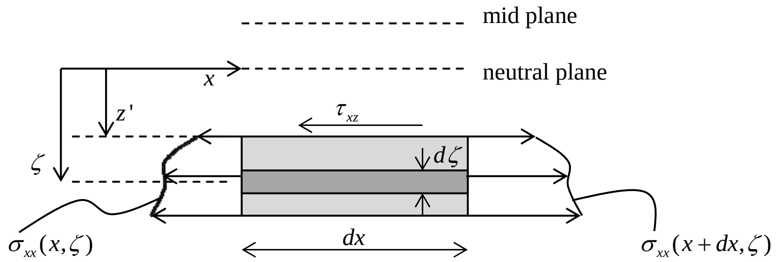

Figure 4: Force equilibrium to evaluate shear stress distribution τ_{xz}

Thus, the shear stress distribution reads

$$\tau_{xz}(z') = \frac{Q_z}{w(z')E_b I} \int_{z'}^{h/2-\bar{z}} E(\zeta)\zeta w(\zeta) d\zeta \,, \tag{39}$$

and the shear correction factor α_s is found from (36) with

$$\frac{1}{\alpha_s} = \frac{\bar{G}A}{\left(E_b I\right)^2} \int_{-h/2-\bar{z}}^{h/2-\bar{z}} \frac{2(1+\nu(z'))}{w(z')E(z')} \left[\int_{z'}^{h/2-\bar{z}} E(\zeta)\zeta w(\zeta) d\zeta \right]^2 dz' \,. \tag{40}$$

3 HOMOGENIZATION OF THE FGM SHELLS

The homogenization procedure for FGM shells with arbitrary distributed material properties can be found directly from the considerations of section (2.3) with $w(z) = 1$. A detailed derivation can be found in [11] and [12].

4 NUMERICAL EXPERIMENTS

The proposed homogenization methods were applied to homogenization for chosen material properties variations. The homogenized effective material properties were then used in multiphysical and modal analysis of the FGM beams and shells. Obtained results confirmed very good effectiveness and accuracy of our approaches. From point of view to meet the maximal number of the contribution pages, the results of provided numerical experiments will be presented by oral presentation of the contribution at the conference WCCM XI.

5 CONCLUSION

Homogenization of the material properties for FGM beams and shells, presented in the contribution, has been done by: (i) the multilayer method by use the extended mixture rules and laminate theory; (ii) by the direct integration method. The disadvantage of the direct integration method by homogenization of transversal continuously varying material properties is that the homogenized effective material properties are obtained via an integration of varying material properties along the height of cross-section, but the results are very accurate. As was shown in [8], [9], [10], any discontinuity in stress and displacements in structural analysis arises. When the variation of material properties is more complex the integration can bring some numerical difficulties. The disadvantage of the multilayer method is that the enough fine discretization in transversal direction on the layers is needed to obtain sufficient solution accuracy. As was shown in [8], the discontinuity of the secondary variables of the structural analysis arises at the layer interfaces. The discontinuity of these variables (e.g. normal and tangential stress, thermal heat flux) can be smoothed out with finer discretization at layers or with a calculation of their average values. But the homogenization method is possible also for more complex systems without any solution problems, and also for the real multilayer beams and shells, where the discontinuity of the secondary variables regularly occurs at the layers interfaces. The homogenization of the varying material properties of the FGM beams in 3 directions (longitudinal, transversal and lateral) will be presented in our future contributions.

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