11th World Congress on Computational Mechanics (WCCM XI)
5th European Conference on Computational Mechanics (ECCM V)
6th European Conference on Computational Fluid Dynamics (ECFD VI)
E. Oñate, J. Oliver and A. Huerta (Eds)

A NEW 3D FGM BEAM FINITE ELEMENT FOR MODAL ANALYSIS

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Key Words: *3D FGM Beam, Spatial Variation of Material Properties, Modal Analysis.*

Abstract. In this contribution, a new 3D-beam finite element of double symmetric cross-sectional area, made of a Functionally Graded Material (FGM) is presented, which can be used in modal, elastostatic and buckling analysis of single beams and beam structures. There, the material properties vary continuously in longitudinal direction while the variation with respect the transversal and lateral directions is assumed to be symmetric in a continuous or discontinuous manner (Figure 1). The shear force deformation effect and the effect of consistent mass distribution and mass moment of inertia are taken into account. Additionally, the Winkler elastic foundation and the effect of axial force are included by the finite element equation derivation as well. Homogenization of the spatially varying material properties to the effective material properties with longitudinal variation is done by multilayer method. For the homogenized beam the finite element matrix, consisting of the stiffness and mass inertia terms, is established. Numerical experiment is made to show the accuracy and effectiveness of the new 3D FGM beam element.

1 INTRODUCTION

Important classes of structural components, where functionally graded material (FGM) is used, are beams and beam structures. FGM beams play important role not only in structural applications, but we can find many applications of the beam structures in thermal, electric-thermal or electric-thermal-structural systems (e.g. MEMS sensors and actuators, and other mechatronic devices). In all these applications, using new materials like FGM can greatly improve efficiency of a system. FGM is built as a mixture of two or more constituents which have almost the same geometry and dimensions. The plasma spraying, powder metallurgy and other technologies are used for fabrication of such materials. From macroscopic point of view, FGM is isotropic in each material point but the material properties can vary continuously or discontinuously in one, two or three directions. The variation of macroscopic material properties can be caused by varying the volume fraction of the constituents or with varying of the constituents material properties (e.g. by non-homogeneous temperature field). Fabrication of such materials is complicated but a development in this area has progressed significantly in recent years. In the literature, many papers deal with static and dynamic analysis of the FGM single 2D beams with only transversal variation of the material properties. The longitudinal stiffness of the beam can be graded with varying cross - sectional area and with varying material properties. In [1], dynamic characteristics of a functionally graded beam with axial or transversal material gradation along the thickness on the power law have been studied. But we did not find in the literature papers which consequently deal with both the longitudinal and transversal variation of material properties whether in dynamics of the single beams neither in the beam structures built of such FGM beams. In [2, 3, 4], we dealt with the calculation of the free vibration of a single 2D FGM beam with continuous spatial polynomial variation of material properties by a fourth-order differential equation of the second order beam theory. The aim of this publication was also to present a new concept for expanding the second order bending beam theory considering the shear deformation according to Timoshenko beam theory. The shear deformation effect in FGM beam with spatial continuous variation of material properties is included here originally by means of the average shear correction factor that has been obtained by an integration of the shear correction function [5]. The continuous polynomial spatial variation of the effective elasticity modulus and mass density can be caused by continuous polynomial spatial variation of both the volume fraction and material properties of the FGM constituents. A choice of the polynomial gradation of material properties enables an easier integration of the derived differential equation and allows to model common variations of material properties. The effect of consistent mass distribution and mass inertia moment and the effect of large axial forces have been taken into account.

The presented contribution is the continuation of our previous work pointed out to derivation of general homogenized 3D FGM beam finite element with the longitudinally polynomial varying effective material properties. Homogenization of the spatial continuously varying material properties in the real FGM beam and the calculation of its other parameters are done by the layering method [6]. If only transversal and lateral variations of material properties are considered in the real FGM beam, the longitudinally constant effective material properties arise from the homogenization. This method can be also used in the homogenization of multilayer beams with discontinuous variation (multilayer beam) of material properties in transversal and lateral direction. The effect of consistent mass distribution and mass inertia moment and the effects of large axial forces and the shear forces can be analyzed, as well. Numerical experiment is performed to calculate the eigenfrequencies of chosen 3D FGM beam of rectangular cross-section with symmetrically lateral and transversal variation of material properties. The solution results are discussed and compared with those obtained using a very fine mesh of the solid finite elements.

2 THE FGM 3D BEAM FINITE ELEMENT EQUATIONS

Let us consider a 3D straight beam finite element of doubly symmetric cross-section – Figure 1. The nodal degrees of freedom at node *i* are: the displacements u_i, v_i, w_i in the local axis direction *x*, *y*, *z*, and the cross-sectional area rotations $\varphi_{x,i}, \varphi_{y,i}, \varphi_{z,i}$. The degrees of freedom at the node *j* are denoted by a similar manner. The internal forces at node *i* are: the axial force N_i, the transversal forces $R_{y,i}$ and $R_{z,i}$, the bending moments $M_{y,i}$ and $M_{z,i}$, and the torsion moment $M_{x,i}$. The internal forces at the node j are denoted by a similar manner. The first derivative after x of the relevant variable is denoted by upper symbol "′".

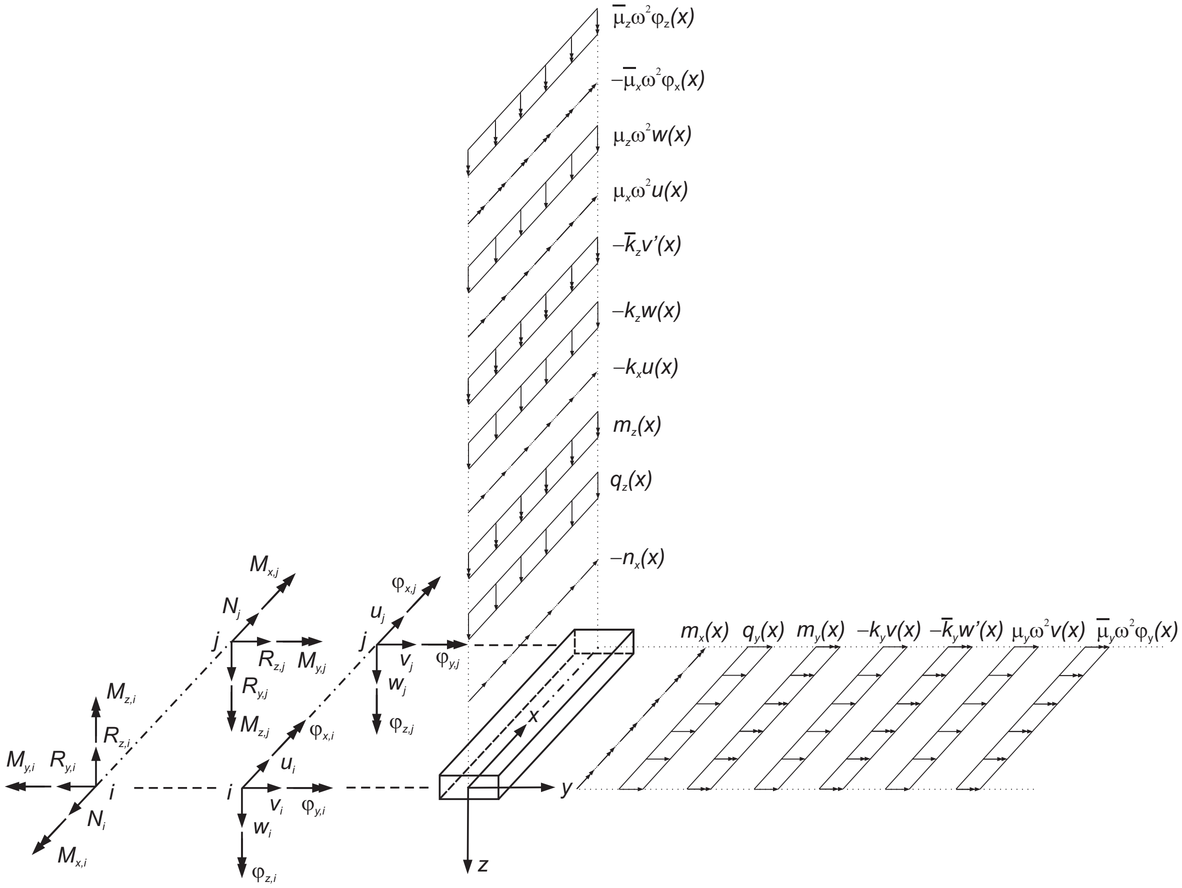

Figure 1: The internal variables and loads

Furthermore, n_x is the axial force distribution, q_z and q_y are the transversal and lateral forces distribution, m_x, m_y and m_z are the distributed moments, $\mu_x = \rho A = \mu_y = \mu_z = \mu$ is the mass distribution, $\bar{\mu}_y = \rho I_y$ and $\bar{\mu}_z = \rho I_z$ and $\bar{\mu}_{xT} = \rho I_p$ are the mass moments of inertia distribution ($\rho = \rho_L^H(x)$ is the homogenized effective mass density distribution, $I_y = bh^3/12$ and $I_z = b^3h/12$ are the quadratic moments of inertia, $I_p = I_y + I_z$ the polar moment of inertia), $k_x, k_y, k_z, \bar{k}_y, \bar{k}_z$ are the elastic foundation modules (the torsional elastic foundation is not considered), and ω is the natural eigenfrequency. The effective homogenized and longitudinally varying stiffness reads: $EA = E_L^{NH}(x)A$ is the axial stiffness ($E_L^{NH}(x)$ is the effective elasticity modulus for axial loading), $EI_y = E_L^{M_yH}(x)I_y$ is the flexural stiffness about axis y ($E_L^{M_yH}(x)$ is the effective elasticity modulus for bending about axis y), $EI_z = E_L^{M_zH}(x)I_z$ is the flexural stiffness in axis z, ($E_L^{M_zH}(x)$ is the effective elasticity modulus

for bending about axis *z*), $G\overline{A}_y = G_{Ly}^{H}(x) k_y^{sm} A$ is the reduced shear stiffness in *y* – direction ($G_{Ly}^{H}(x)$ is the shear modulus and k_y^{sm} is the average shear correction factor in *y* – direction), $G\overline{A}_z = G_{Lz}^{H}(x) k_z^{sm} A$ is the reduced shear stiffness in *z* – direction ($G_{Lz}^{H}(x)$ is the shear modulus and k_z^{sm} is the average shear correction factor in *z* – direction), $G_L^{M_x H}(x) I_T$ is the torsional stiffness ($G_L^{M_x H}(x)$ is the torsional elasticity modulus ant I_T is the torsion constant).
For establishing of the FGM 3D beam finite element equation we will use following differential equations for axial, transversal, lateral and torsional free vibration (according the Figure 1).

2.1 Axial free vibration

By combination of the main equations for the axial vibration (1) and (2) of the FGM beam

$$N' = n_x + \left(k_x - \mu_x \omega^2 u\right) \tag{1}$$

$$u' = \frac{N}{EA} \tag{2}$$

we get the differential equation with non-constant polynomial coefficients (3)

$$\eta_{2u} u'' + \eta_{1u} u' + \eta_{0u} u = n \tag{3}$$

where $\eta_{2u} = EA,\ \eta_{1u} = E'A,\ \eta_{0u} = \mu\omega^2 - k_x$. In the modal axial vibration analysis the right side of the equations (3) is equal to zero.
Here, n_x is the axial distributed load; N=$N(x)$ is the axial force and N' is its first derivative; $k_x = k_x(x)$ is the modulus of elastic foundation in the axial direction; $u = u(x)$ is the axial displacement and $u' = u'(x)$ is its first derivative; $EA = E_L^{NH}(x) A$ is the homogenized beam stiffness in axial direction, E' is the first derivative of $E_L^{NH}(x)$ and ω is the natural frequency.
The solution of (3) can be done by transfer functions $\overline{b}_{jN}$ for axial loading [7, 8]:

$$\begin{bmatrix} u(x) \\ u'(x) \end{bmatrix} = \begin{bmatrix} \overline{b}_{0N} & \overline{b}_{1N} \\ \overline{b}'_{0N} & \overline{b}'_{1N} \end{bmatrix} \cdot \begin{bmatrix} u_i \\ u'_i \end{bmatrix} \tag{4}$$

Here, u_i is the axial displacement and u'_i is its value of the first derivative at node *i*.

If the $u'(x)$ and u'_i are replaced with the constitutive equation of the FGM beam $u' = \dfrac{N}{EA}$, we get:

$$\begin{bmatrix} u(x) \\ N(x) \end{bmatrix} = \begin{bmatrix} \overline{b}_{0N} & \dfrac{\overline{b}_{1N}}{E_i A} \\ EA\overline{b}'_{0N} & \dfrac{EA}{E_i A}\overline{b}'_{1N} \end{bmatrix} \cdot \begin{bmatrix} u_i \\ N_i \end{bmatrix} \tag{5}$$

where E_i is the initial value of the homogenized elasticity modulus $E_L^{NH}(x)$ at node *i*. By setting $x = L$ in (11) the dependence of the nodal variables at node *j* on the nodal variables at node *i* will be obtained. By appropriated mathematical operation the local finite element equation for axial free vibration were obtained:

$$\begin{bmatrix} N_i \\ N_j \end{bmatrix} = \begin{bmatrix} B_{1,1} & B_{1,7} \\ B_{7,1} & B_{7,7} \end{bmatrix} \cdot \begin{bmatrix} u_i \\ u_j \end{bmatrix} \tag{6}$$

The stiffness constants $B_{i,j}$ are calculated numerically by Mathematica [9].

2.2 Flexural free vibration

Homogeneous differential equation of the 4th order with non-constant coefficients of the homogenized FGM beam flexural free vibration (about the *y*-axis) (Figure 1) has a form (7)

$$\eta_{4w} w'''' + \eta_{3w} w''' + \eta_{2w} w'' + \eta_{1w} w' + \eta_{0w} w = 0 \tag{7}$$

Here $w = w(x)$ is the deflection curve in the $x - z$ plane. Its derivatives after *x* are denoted by the upper symbol.

Derivation of the non-constant coefficients η_{0w} to η_{4w} and appropriated parameters of the differential equation (7) from the main coupled equations (8) and (9) of the 2nd order beam theory (including the inertia forces, shear force and axial force) using the relation between the transversal and shear force (10) is described in [3, 5].

$$R_z' = -q_z + k_z w - \mu\omega^2 w \qquad M_y' = Q_z + m_y + \overline{\mu_y}\omega^2\varphi_y \tag{8}$$

$$\varphi_y' = -\frac{M_y}{EI_y} \Rightarrow M_y = -EI_y\varphi_y' - EI_y\kappa_y^{\ e} \qquad w' = \varphi_y + \frac{Q_z}{G\overline{A}_z} \Rightarrow Q_z = G\overline{A}_z w' - G\overline{A}_z\varphi_y \tag{9}$$

$$Q_z = -\left(\overline{k_z} + N^{II}\right)w' - N^{II}\psi_y + R_z \tag{10}$$

Here, q_z is the distributed transversal load; m_y is the distributed bending moment; κ_y^e is he applied beam curvature; k_z is the modulus of elastic Winkler foundation; μ is the mass distribution; $\overline{\mu}_y$ is the mass inertial moment distribution; ω is the natural eigenfrequency; R_z is the transversal force; Q_z is the shear force; M_y is the bending moment; φ_y is the angle of cross-section rotation; *w* is the beam deflection; EI_y is the bending stiffness and $G\overline{A}_z$ is the reduced shear stiffness of the homogenized FGM beam. $N^{II} \equiv N$ is the resultant axial force of the 2nd order beam theory, ψ_y is the beam rotation imperfection, and $\overline{k}_y$ is the elastic foundation modulus for flexural beam rotation. We assume that all the above quantities are the polynomial functions of *x*. The first derivative after *x* of the respective function is denoted by superscript symbol "′". For the modal analysis, the external loads are equal to zero.

If the variation of the beam parameters is polynomial, the solution of this differential equation based on transfer function [7] has a form:

$$\begin{bmatrix} w(x) \\ w'(x) \\ w''(x) \\ w'''(x) \end{bmatrix} = \begin{bmatrix} b_{0w} & b_{1w} & b_{2w} & b_{3w} \\ b'_{0w} & b'_{1w} & b'_{2w} & b'_{3w} \\ b''_{0w} & b''_{w1} & b''_{2w} & b''_{3w} \\ b'''_{0w} & b'''_{1w} & b'''_{2w} & b'''_{3w} \end{bmatrix} \cdot \begin{bmatrix} w_i \\ w'_i \\ w''_i \\ w'''_i \end{bmatrix} \tag{11}$$

where b_{jw}, b'_{jw}, b''_{jw} and b'''_{jw} functions, ($j \in \langle 0,3 \rangle$), are the solution functions (called transfer functions) of the differential equation (7). The dependence of the $w' = w'(x)$, $w'' = w''(x)$ and $w''' = w'''(x)$ on the $\varphi_y = \varphi_y(x)$, $M_y = M_y(x)$ and $R_z = R_z(x)$ is described in [3] from which the transfer matrix expression has been obtained:

$$\begin{bmatrix} w(x) \\ \varphi_y(x) \\ M_y(x) \\ R_z(x) \end{bmatrix} = \begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} \\ A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} \\ A_{3,1} & A_{3,2} & A_{3,3} & A_{3,4} \\ A_{4,1} & A_{4,2} & A_{4,3} & A_{4,4} \end{bmatrix} \cdot \begin{bmatrix} w_i \\ \varphi_{y_i} \\ M_{y_i} \\ R_{zi} \end{bmatrix} \quad ()$$

The $A_{i,j}$ functions are evaluated by software Mathematica [9], and they are of relative complicated form. The kinematical and kinetic variables at node *i* are denoted by index *i* in (). By setting $x = L$ in (12) the dependence of the nodal variables at node *j* on the nodal variables at node *i* will be obtained.

By appropriated mathematical operation the local finite element equation for flexural free vibration (about the *y* – axis) were obtained:

$$\begin{bmatrix} R_{z,i} \\ M_{y,i} \\ R_{z,j} \\ M_{y,j} \end{bmatrix} = \begin{bmatrix} B_{3,3} & B_{3,6} & B_{3,9} & B_{3,12} \\ B_{5,2} & B_{5,5} & B_{5,8} & B_{5,11} \\ B_{9,3} & B_{9,6} & B_{9,9} & B_{9,12} \\ B_{11,2} & B_{11,5} & B_{11,8} & B_{11,11} \end{bmatrix} \cdot \begin{bmatrix} w_i \\ \varphi_{y,i} \\ w_j \\ \varphi_{y,j} \end{bmatrix} \tag{13}$$

The stiffness constants $B_{i,j}$ are calculated numerically by Mathematica.

Homogeneous differential equation of the 4th order with non-constant coefficients of the homogenized FGM beam flexural free vibration (about the *z*-axis), (Figure 1), can be derived similarly to the previous case, and it has a form (14)

$$\eta_{4v}v'''' + \eta_{3v}v''' + \eta_{2v}v'' + \eta_{1v}v' + \eta_{0v}v = 0 \tag{14}$$

By appropriated mathematical operation the local finite element equation for flexural free vibration (about the *z* – axis) were obtained:

$$\begin{bmatrix} R_{y,i} \\ M_{z,i} \\ R_{y,j} \\ M_{z,j} \end{bmatrix} = \begin{bmatrix} B_{2,2} & B_{2,6} & B_{2,8} & B_{2,12} \\ B_{6,2} & B_{6,6} & B_{6,8} & B_{6,12} \\ B_{8,2} & B_{8,6} & B_{8,8} & B_{8,12} \\ B_{12,2} & B_{12,6} & B_{12,8} & B_{12,12} \end{bmatrix} \cdot \begin{bmatrix} v_i \\ \varphi_{z,i} \\ v_j \\ \varphi_{z,j} \end{bmatrix} \tag{15}$$

The stiffness constants $B_{i,j}$ are calculated numerically by Mathematica [9].

2.3 Torsional free vibration

The differential equations of uniform torsion free vibration of a beam are formulated according the Figure 1 and has a form:

$$M'_x = m_x - \rho_L^H I_p \omega^2 \varphi_x \tag{16}$$

$$\varphi'_x = \frac{M_x}{G_L^{M_x H} I_T} \tag{17}$$

There, I_p is the cross-sectional area polar moment of inertia; I_T is the torsion constant ($I_p = I_T$ for the circular cross-section); ω is the natural frequency; φ_x is the torsion angle of rotation; φ'_x is a first derivative of the torsion angle. The load – distributed torsion moment – is denoted by $m_x = m_x(x)$. The derivative of appropriate variable is denoted by “′”

By combination of equations (16) and (17) and after some mathematical manipulations the differential equation has been obtained:

$$\eta_{2T}\varphi''_x + \eta_{1T}\varphi'_x + \eta_{0T}\varphi_x = m_x \tag{18}$$

with non-constant parameters $\eta_{1T} = G_L^{M_x H} I_T$; $\eta_{2T} = G_L'^{M_x H} I_T$; $\eta_{0T} = \rho_L^H I_p \omega^2$. According to [7], the solution of the differential equation (18) is:

$$\begin{bmatrix} \varphi_x \\ \varphi'_x \\ 1 \end{bmatrix} = \begin{bmatrix} \overline{b}_{0T} & \overline{b}_{1T} & \varphi_p \\ \overline{b}'_{0T} & \overline{b}'_{1T} & \varphi'_p \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \varphi_{x,i} \\ \varphi'_{x,i} \\ 1 \end{bmatrix} \tag{19}$$

There $\overline{b}_{jT}$ and $\overline{b}'_{jT}$, $j \in \langle 0,1 \rangle$, are the transfer functions (of x) and their first derivatives, respectively. Those are the solution functions of the differential equation (18). The transfer functions depend on the longitudinal variation of the torsional shear modulus, the natural frequency, the polar moment of inertia, the torsion constant and the consistent mass density distribution. The φ_p and φ'_p represents the influence of distributed loading on torsion angle of rotation and the first derivative of torsion angle and will be calculated from the inhomogeneous solution of the differential equation. In modal analysis these terms are equal to zero. By setting (16) and (17) into the (19) and by some mathematical manipulations, the transfer matrix relations (20) for uniform torsion have been obtained:

$$\begin{bmatrix} \varphi_x \\ M_x \\ 1 \end{bmatrix} = \begin{bmatrix} \overline{b}_{0T} & \dfrac{\overline{b}_{1T}}{G_{Li}^H I_T} & \varphi_p \\ G_L^{M_x H} I_T \overline{b}'_{0T} & \dfrac{G_L^{M_x H} I_T}{G_{Li}^{M_x H} I_T} \overline{b}'_{1T} & G_L^{M_x H} I_T \varphi'_p(x) \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \varphi_{x,i} \\ M_{x,i} \\ 1 \end{bmatrix} \tag{20}$$

By setting $x = 0$ resp. $x = L$ in (20) a dependence of state variables at point j on the state variables at the initial point i for modal analysis has been obtained:

$$\begin{bmatrix} \varphi_{x,j} \\ M_{x,j} \end{bmatrix} = \begin{bmatrix} \left[\bar{b}_0\right]_{x=L} & \dfrac{\left[\bar{b}_1\right]_{x=L}}{G_{Li}^{M_xH} I_T} \\ G_{Lj}^{M_xH} I_T \left[\bar{b}_0'\right]_{x=L} & \dfrac{G_{Lj}^{M_xH} I_T}{G_{Li}^{M_xH} I_T} \left[\bar{b}_1'\right]_{x=L} \end{bmatrix} \cdot \begin{bmatrix} \varphi_{x,i} \\ M_{x,i} \end{bmatrix}. \tag{21}$$

By simple mathematical manipulation we get the local finite element equation:

$$\begin{bmatrix} M_{x,i} \\ M_{x,j} \end{bmatrix} = \begin{bmatrix} B_{4,4} & B_{4,10} \\ B_{10,4} & B_{10,10} \end{bmatrix} \cdot \begin{bmatrix} \varphi_{x,i} \\ \varphi_{x,j} \end{bmatrix}, \tag{22}$$

with the finite element matrix terms: $B_{4,4} = \dfrac{\left[\bar{b}_0\right]_{x=L}}{\left[\bar{b}_1\right]_{x=L}} G_{Li}^{M_xH} I_T$, $B_{4,10} = -\dfrac{G_{Li}^{M_xH} I_T}{\left[\bar{b}_1\right]_{x=L}}$, $B_{10,4} = B_{4,10}$, $B_{10,10} = \dfrac{\left[\bar{b}_1'\right]_{x=L}}{\left[\bar{b}_1\right]_{x=L}} G_{Lj}^{M_xH} I_T$. $\left[\bar{b}_j\right]_{x=L}$ and $\left[\bar{b}_j'\right]_{x=L}$, $j \in \langle 0,1 \rangle$, are the transfer constants which can be calculated by simple numerical algorithm [7, 8]. $G_{Lj}^{M_xH}$ is the value of the homogenized torsional shear modulus at point *j*.

2.4 Local FGM beam finite element equation

The local finite element equation is obtained by superposition of the axial, flexural and torsional free vibration equations of the FGM beam, and it reads:

$$\begin{bmatrix} N_i \\ R_{y,i} \\ R_{z,i} \\ M_{x,i} \\ M_{y,i} \\ M_{z,i} \\ N_j \\ R_{y,j} \\ R_{z,j} \\ M_{x,j} \\ M_{y,j} \\ M_{z,j} \end{bmatrix} = \begin{bmatrix} B_{1,1} & 0 & 0 & 0 & 0 & 0 & B_{1,7} & 0 & 0 & 0 & 0 & 0 \\ & B_{2,2} & 0 & 0 & 0 & B_{2,6} & 0 & B_{2,8} & 0 & 0 & 0 & B_{2,12} \\ & & B_{3,3} & 0 & B_{3,5} & 0 & 0 & 0 & B_{3,9} & 0 & B_{3,11} & 0 \\ & & & B_{4,4} & 0 & 0 & 0 & 0 & 0 & B_{4,10} & 0 & 0 \\ & S & & & B_{5,5} & 0 & 0 & 0 & B_{5,9} & 0 & B_{5,11} & 0 \\ & & M & & & B_{6,6} & 0 & B_{6,8} & 0 & 0 & 0 & B_{6,12} \\ & & & M & & & B_{7,7} & 0 & 0 & 0 & 0 & 0 \\ & & & & E & & & B_{8,8} & 0 & 0 & 0 & B_{8,12} \\ & & & & & T & & & B_{9,9} & 0 & B_{9,11} & 0 \\ & & & & & & R & & & B_{10,10} & 0 & 0 \\ & & & & & & & Y & & & B_{11,11} & 0 \\ & & & & & & & & & & & B_{12,12} \end{bmatrix} \cdot \begin{bmatrix} u_i \\ v_i \\ w_i \\ \varphi_{x,i} \\ \varphi_{y,i} \\ \varphi_{z,i} \\ u_j \\ v_j \\ w_j \\ \varphi_{xj} \\ \varphi_{yj} \\ \varphi_{zj} \end{bmatrix} \tag{23}$$

In (23) the $B_{i,j}$ are the terms of the local beam element matrix. In modal analysis of the single straight beam the global finite element matrix coincides with the local matrix. The global matrix of the beam structures can be established by a usual method. In modal analysis the eigenvalue problem is solved. By given geometrical, material and boundary conditions the natural frequency ω will be increased until the determinant of the global matrix is zero. The natural frequency is the natural eigenfrequency from which the eigenfrequency can be calculated.

3 HOMOGENIZATION OF SPATIAL VARYING MATERIAL PROPERTIES FOR THE 3D BEAM APPLICATIONS BY THE MULTILAYERING METHOD

Let us consider a two nodal 3D straight beam element with predominantly rectangular cross-sectional area *A*. The composite material of this beam arises from mixing two components (matrix and fibres) that are approximately of the same geometrical form and dimensions. The continuous polynomial spatial variation of the elasticity moduli and mass density (only transversal and lateral variations of material properties are considered) can be caused by continuous polynomial spatial variation of both the volume fraction (fibres - $v_f(y,z)$ and matrix - $v_m(y,z)$) and material properties of the FGM constituents (fibres - $p_f(y,z)$ and matrix $p_m(y,z)$).

Homogenization of the material properties (the reference volume is the volume of the whole beam) will be done in two steps by the multilayer method. In the first step, the real beam (Figure 2a) will be transformed into a multilayer beam (Figure 2b). Material properties of the layers will be calculated via the extended mixture rules [5]. Each layer will have constant volume fractions and material properties. In the second step, the effective longitudinal material properties of the homogenized beam will be derived using the laminate theory. These homogenized material properties are constant through the beam height and width (Figure 1c).

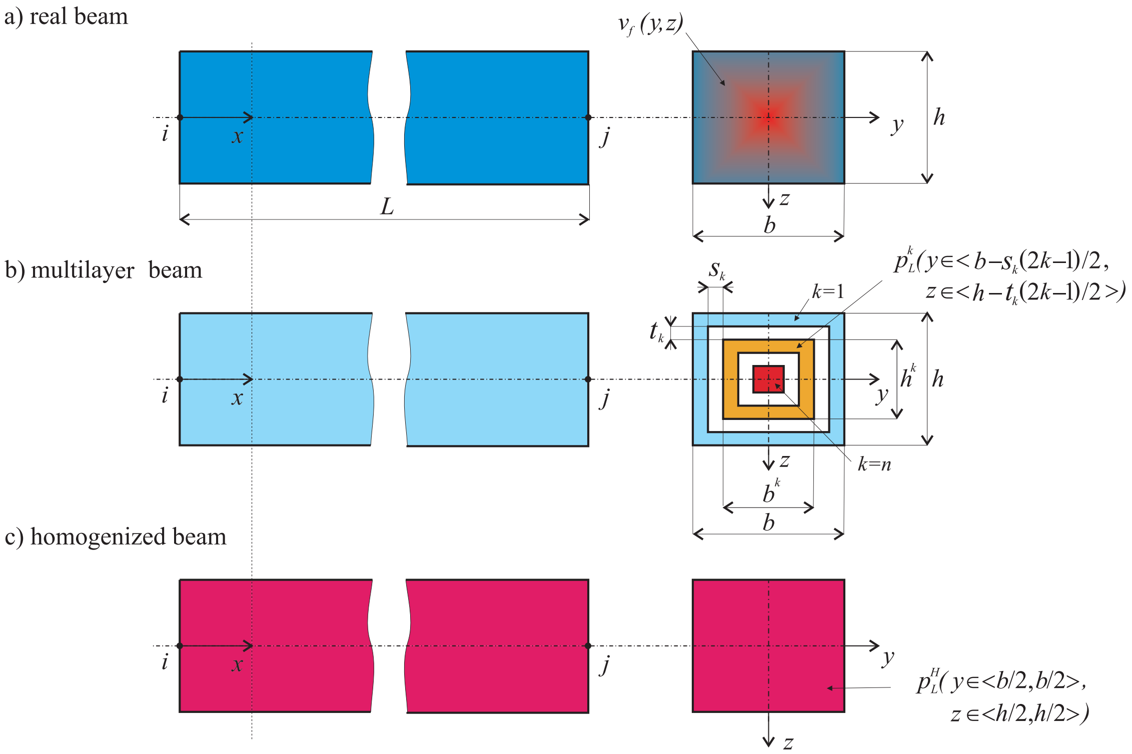

Figure 2: Homogenization of material properties.

The material properties in the real beam can be calculated using extended mixture rule.

$$p(y,z) = v_f(y,z)p_f(y,z) + v_m(y,z)p_m(y,z) \tag{24}$$

In our case the elasticity modulus $E(x,y)$, Poisson ratio $\nu(x,y)$, and mass density $\rho(x,y)$ have been calculated by expression (24). The FGM shear modulus can be calculated by expression:

$$G(y,z) = \frac{E(y,z)}{2(1+\nu(y,z))} \tag{25}$$

In the homogenization of the spatial varying material properties (24) - (25) the multilayering method will be used. The homogenized elasticity moduli for: tension-compression - E_L^{NH}, bending about axis y - $E_L^{M_yH}$, bending about axis z - $E_L^{M_zH}$, shear in y direction - G_{Ly}^H, shear in z direction - G_{Lz}^H, the torsional elasticity modulus $G_L^{M_xH}$ and the homogenized mass density ρ_L^H can be calculated, respectively:

$$E_L^{NH} = \frac{\sum_{k=1}^{n} E_k A_k}{A} \qquad E_L^{M_yH} = \frac{\sum_{k=1}^{n} E_k I_{yk}}{I_y} \qquad E_L^{M_zH} = \frac{\sum_{k=1}^{n} E_k I_{zk}}{I_z} \tag{26}$$

$$G_{Ly}^{H} = \frac{\sum_{k=1}^{n} k_{y,k}^{sm} G_k A_k}{k_y^{sm} A} \qquad G_{Lz}^{H} = \frac{\sum_{k=1}^{n} k_{z,k}^{sm} G_k A_k}{k_z^{sm} A} \qquad G_L^{M_xH} = \frac{\sum_{k=1}^{n} G_k I_{Tk}}{I_T} \tag{27}$$

$$\rho_L^H = \frac{\sum_{k=1}^{n} \rho_k A_k}{A} \tag{28}$$

Here, A_k is the cross-section, E_k is the constant elasticity modulus, I_{yk} and I_{zk} are the quadratic moments of inertia, G_k is the constant shear modulus, I_{Tk} is the torsion constant and ρ_k is the constant mass density of the k^{th} layer. The exact expression for homogenization of spatial varying (continuously or discontinuously and symmetrically in transversal and lateral direction, and continuously in longitudinal direction) material properties for the FGM beams of selected doubly-symmetric cross-sections will be presented in [10] in detail.

4 NUMERICAL EXPERIMENT

The clamped FGM beam has been considered (as shown in Figure 3). Its rectangular cross-section is constant with height h = 0.005 m and width b = 0.01 m. Length of the beam is L = 0.1 m. Material of the beam consists of two components: aluminum Al6061-TO – as a matrix and denoted with index m; titanium carbide TiC – as a fibre and denoted with index f.

Material properties of the components are assumed to be constant and their values are: aluminum Al6061-TO (matrix) – the elasticity modulus $E_m = 69.0$ GPa, the mass density $\rho_m = 2700$ kgm^{-3}, the Poisson's ratio $\nu_m = 0.33$; titanium carbide TiC (fibres) – the elasticity modulus $E_f = 480.0$ GPa, the mass density $\rho_f = 4920$ kgm^{-3}, the Poisson's ratio $\nu_f = 0.20$.

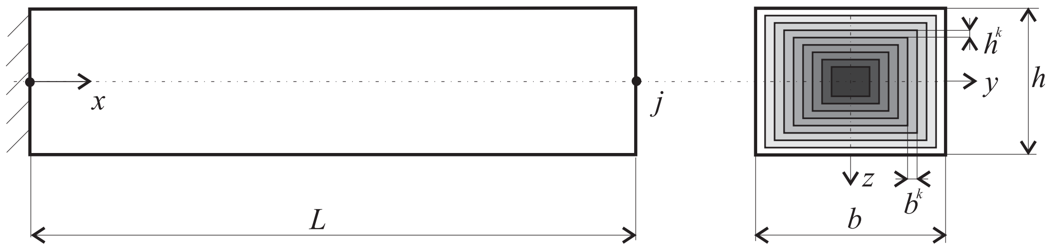

Figure 3: FGM beam with spatial variation of material properties

The fibre volume fraction varies linearly and symmetrically according to the *y*-*z* plane $v_f \in \langle 0; 1.0 \rangle$ - the core of the beam is made from pure matrix and linearly vary to the edges that are made from pure fibre. The average shear correction factor in *y* – direction $k_y^{sm} = 5/6$ and in *z* – direction $k_z^{sm} = 5/6$ have been considered (constant Poisson ratio has been assumed for simplicity).

Using multilayer method the effective elasticity modulus for axial loading E_L^{NH} in [GPa], for bending about axis *y* - $E_L^{M_y H}$ and about axis *z* - $E_L^{M_z H}$ in [GPa], shear moduli G_{Ly}^H and G_{Lz}^H in [GPa], torsional shear modulus $G_L^{M_x H}$ in [GPa] and mass density ρ_L^H in [kgm^{-3}] have been calculated:

$$E_L^{NH} = 342.11\,\text{GPa}; \qquad E_L^{M_y H} = E_L^{M_z H} = 396.43\ \text{GPa};$$

$$G_{Ly}^H = G_{Lz}^H = 138.58\ \text{GPa}; \qquad G_L^{M_x H} = 162.23\ \text{GPa} \qquad \rho_L^H = 4175.19\ \text{kgm}^{-3}$$

The FGM beam clamped at the node *i* has been studied by modal analysis. The first eight eigenfrequencies *f* [Hz] have been found (see Table 1) using the new FGM beam finite element (calculation has been done with software Mathematica [9]). Only one our new finite element was used. The same problem has been solved using a very fine mesh – 8967 of SOLID186 elements of the FEM program ANSYS [11]. The average relative difference Δ [%] between eigenfrequencies calculated by our method and the ANSYS solution has been evaluated.

Table 1: Eigenfrequencies of the FGM beam

eigenfrequencies *f* [Hz]		New Finite Element	Ansys	Δ [%]
1st	flexural about axis *y*	785.4	787.1	0.22
2nd	flexural about axis *z*	1560.8	1562.0	0.08
3rd	flexural about axis *y*	4859.9	4864.9	0.10
4th	flexural about axis *z*	9324.5	9297.3	0.29
5th	torsional	11553.0	11213.0	3.03
6th	flexural about axis *y*	13345.6	13340.0	0.04
7th	axial	22630.1	22671.0	0.18
8th	flexural about axis *z*	24228.9	24259.0	0.12

5 CONCLUSION

Originally, new 3D beam finite element for modal analysis of the FGM beam structures has been established in the proposed contribution. The obtained results have been studied and compared with results obtained using a very fine mesh of the FEM program ANSYS. As can be observed, an excellent agreement of both solution results has been obtained, which confirmed respectable accuracy and effectiveness of our approach.

Acknowledgement: This paper has been supported by Grant Agency APVV-0246-12, Grant Agency VEGA No. 1/0534/12 and Agency KEGA No. 015STU-4/2012.

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