11th World Congress on Computational Mechanics (WCCM XI)
5th European Conference on Computational Mechanics (ECCM V)
6th European Conference on Computational Fluid Dynamics (ECFD VI)
E. Oñate, J. Oliver and A. Huerta (Eds)

FINITE ELEMENT PHASE-FIELD MODELLING OF BRITTLE FRACTURE

HUGO A.F.A. SANTOS AND VADIM V. SILBERSCHMIDT

* Wolfson School of Mechanical and Manufacturing Engineering
Loughborough University
Loughborough, Leicestershire LE11 3TU, U.K.
e-mails: hugofreixialsantos@gmail.com, v.silberschmidt@lboro.ac.uk

Key words: Brittle Fracture, Phase-Field Model, Finite Element Method, Crack Initiation, Crack Propagation

Abstract. A finite element phase-field model for the analysis of brittle fracture problems is formulated and numerically implemented. The model relies on recently developed strategies for incorporating an additional phase-field to embody fracture. The spatial discretization is formulated by means of the classical Galerkin method, whereas an implicit Euler method with adaptive time-stepping is adopted for the temporal discretization. To demonstrate the capabilities of the model, a two-dimensional square plate under tensile loading is modelled. The obtained results show that, as expected, the stress singularities at the corners of the plate are responsible for the crack initiation and propagation.

1 INTRODUCTION

The prevention of fracture-induced failure is a major constraint in engineering design processes, and the numerical simulation of fracture processes often plays a key role in design decisions.

Although a huge effort has been made to develop novel and accurate models of fracture and an enormous progress has been achieved, the development of an adequate scheme for the numerical simulation of crack initiation and propagation is still a significant challenge for the Computational Mechanics community. A particularly successful approach is provided by the Linear Elastic Fracture Mechanics (LEFM) theory, based on Griffith's theory for brittle fracture, which relates crack nucleation and propagation to a critical value of the energy release rate. However, this theory suffers from several deficiencies, namely: (i) inability to predict crack initiation (local stress needs to be infinite); (ii) inadequacy to predict the direction of a crack path; (iii) inability to handle crack jumps; and (iv) inability to predict crack branching.

From a numerical point of view, the efforts to model brittle fracture have been focused essentially on two broad approaches: (i) discrete methods, such as the element deletion method, the embedded finite element method or the extended finite element method, which use the FEM in conjunction with Griffith's-type LEFM models to incorporate discontinuities into the displacement field, and (ii) continuum-damage (CD) methods, which incorporate a damage parameter into the model that describes the material's deterioration and controls its strength. Some of these methods are already available in commercial CAE software packages and can be used for various applications. However, it has long been recognized that, while the discrete methods are well suited only for static fracture and when a moderate number of cracks occurs, the CD methods are not effective when modelling large dominant cracks, since the damage zone tends to widen in a direction normal to the crack initiation as the simulation proceeds. Another shortcoming of the CD methods is that regularization algorithms are needed to overcome mesh dependency.

Additionally, current methods for predicting crack propagation, in particular for dynamics and 3D problems, still lack accuracy and robustness, even when applied to relatively simple benchmark tests [1].

Due to these reasons, several efforts were made in the last decade to develop alternative schemes. Recently, a new approach for the numerical simulation of fracture has emerged. In this approach, discontinuities are not introduced into the solid to represent cracks. Instead, a fracture locus is approximated by a phase-field, which smoothes the boundary of a crack over a small region [4, 2, 3]. The major advantage of using the phase-field approach is that the evolution of fracture surfaces follows from the solution of a coupled system of partial differential equations. As opposed to many discrete methods, its implementation does not require fracture surfaces to be tracked algorithmically. Phase-field models avoid these difficulties by introducing a continuous phase-field variable which smoothly interpolates between undamaged and fully failed states of the modelled material.

The goal of this paper is to present a finite element phase-field model for brittle fracture that is capable of reproducing various complex phenomena, such as branching of pre-existing cracks, as well as the nucleation of new cracks in originally undamaged domains.

2 PHASE-FIELD MODEL OF BRITTLE FRACTURE

Consider an arbitrary body $\Omega \in \mathcal{R}^2$ with external boundary $\partial\Omega$ and internal discontinuity boundary Γ. Let $\partial\Omega = \partial\Omega_N \cup \partial\Omega_D$, with $\partial\Omega_N$ and $\partial\Omega_D$ being the Neumann and Dirichlet boundaries, respectively, such that $\partial\Omega_N \cap \partial\Omega_D = \emptyset$. The displacement of a point $\boldsymbol{x} \in \Omega$ at time t is denoted by $\boldsymbol{u}(\boldsymbol{x}, t)$, and $\boldsymbol{\varepsilon}(\boldsymbol{x}, t)$ is its associated infinitesimal strain vector. $\boldsymbol{u}$ may be a result of either the applied body forces $\boldsymbol{b}$, tractions $\bar{\boldsymbol{t}}$, prescribed boundary displacements $\bar{\boldsymbol{u}}$, or a combination of these actions. The body is assumed linear elastic, homogeneous, and isotropic with Young's modulus E and Poisson's ratio ν.

The Lagrange energy functional of the problem is taken as

$$L(\boldsymbol{u}, \dot{\boldsymbol{u}}, c) = \int_\Omega \left(\frac{1}{2}\rho \dot{\boldsymbol{u}} \cdot \dot{\boldsymbol{u}} - \frac{1}{2}(c^2 + \eta)\boldsymbol{\varepsilon}^T \boldsymbol{C} \boldsymbol{\varepsilon} \right) d\Omega - \int_\Omega \mathcal{G}_c \left(\frac{(1-c)^2}{4\epsilon} + \epsilon \boldsymbol{\nabla} c \right) d\Omega \qquad (1)$$

where $\dot{\boldsymbol{u}}$ stands for the derivative of $\boldsymbol{u}$ with respect to time, c is the phase-field, $\mathcal{G}_c$ is the energy release rate, ϵ is the regularization parameter, η is the viscosity parameter and $\boldsymbol{C}$ stands for the standard constitutive operator that maps infinitesimal strains, $\boldsymbol{\varepsilon}$, onto Cauchy stresses, $\boldsymbol{\sigma}$, in the two-dimensional space.

c takes values from 0 to 1. $c = 0$ inside the crack and $c = 1$ away from the crack. ϵ controls the width of the smooth approximation of the crack and η is a small dimensionless parameter ($0 < \eta \ll 1$) introduced to avoid numerical difficulties where the material is totally damaged (*i.e.*, where $c = 0$).

The Euler-Lagrange equations of the problem can be used to arrive at the following strong form of the equations of motion

$$\boldsymbol{D}^T\boldsymbol{\sigma} + \boldsymbol{b} = \rho\ddot{\boldsymbol{u}} \tag{2a}$$

$$\dot{c} = -M\left(c\boldsymbol{\varepsilon}^T\boldsymbol{C}\boldsymbol{\varepsilon} - \mathcal{G}_c\left(2\epsilon\Delta c + \frac{1-c}{2\epsilon}\right)\right) \tag{2b}$$

with

$$\boldsymbol{\sigma} = (c^2 + \eta)\boldsymbol{C}\boldsymbol{\varepsilon}$$

M is a mobility parameter and is assumed in this work to be constant and positive. As a result, the adopted phase-field evolution model represents the classical Ginzburg-Landau model [4].

Additionally, the following sets of boundary conditions

$$\boldsymbol{u} = \bar{\boldsymbol{u}}, \text{ on } \partial\Omega_D$$
$$\mathbf{N}\boldsymbol{\sigma} = \bar{\boldsymbol{t}}, \text{ on } \partial\Omega_N$$

with

$$\mathbf{N} = \begin{bmatrix} n_1 & 0 & n_2 \\ 0 & n_2 & n_1 \end{bmatrix}$$

where n_j are the components of the external unit normal $\boldsymbol{n}$ of $\partial\Omega_N$, and initial conditions

$$\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x}), \ \boldsymbol{x} \in \Omega$$
$$\dot{\boldsymbol{u}}(\boldsymbol{x}, 0) = \boldsymbol{v}_0(\boldsymbol{x}), \ \boldsymbol{x} \in \Omega$$

must hold. $\bar{\boldsymbol{u}}$ and $\bar{\boldsymbol{t}}$ are the prescribed displacements on $\partial\Omega_D$ and applied tractions on $\partial\Omega_N$, respectively. $\boldsymbol{u}_0$ and $\boldsymbol{v}_0$ are the displacements and velocities, respectively, for $t = 0$.

Perhaps the most significant drawback of variationally-based phase-field models is that they typically allow for crack healing. To rule out such an unphysical behavior, Hakim and Karma propose constraining the time-rate of the phase field to be less than or equal to zero at all points and times [4]. Alternatively, following the approach proposed in [5], the irreversibility of cracking can be guaranteed by imposing homogeneous Dirichlet boundary conditions on the phase-field once a crack is detected. The latter is the formulation adopted in this work.

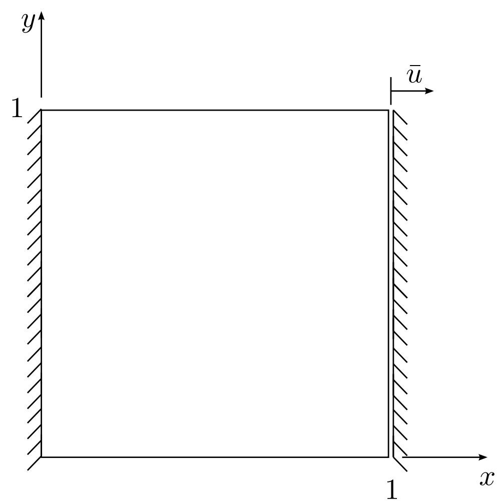

Figure 1: Square plate under quasi-static tensile load: problem definition

3 FINITE ELEMENT MODEL

The implementation of the initial boundary-value problem into a finite element scheme requires both a spatial and temporal discretization. In this work, the spatial discretization is formulated by means of the classical Galerkin method, whereas an implicit Euler method with adaptive time-stepping is adopted for the temporal discretization of the transient evolution equation. In particular, for the discretization in space, four-node quadrilateral (bilinear Lagrangian) elements with three degrees-of-freedom per node, *i.e.*, two displacement and one phase-field degrees-of-freedom per node, and a 2×2 Gauss quadrature rule are adopted in this work. The nonlinear coupled system of equations is solved using a Newton-Raphson algorithm.

4 NUMERICAL RESULTS

4.1 Square plate under quasi-static prescribed displacement

The square plate under quasi-static prescribed displacement depicted in Figure 1 was considered. A uniform 64×64 mesh was adopted. The material parameters (assumed to be dimensionally consistent) were set to $E = 10^6$, $\nu = 0.3$ and $\mathcal{G}_c = 0.01$, and plane stress was assumed. The regularization, viscosity and mobility parameters were chosen to be $\epsilon = 0.02$, $\eta = 10^{-5}$ and $M = 10^9$, respectively. The prescribed displacement was set to $\bar{u} = 4.5 \times 10^{-4}$. The problem was solved in an incremental-iterative fashion by resorting to the Newton-Raphson method. An adaptive time-stepping scheme was used, for which the convergence tolerance was set to 1.0×10^{-9} and the optimal number of iterations to 4. For a larger number of iterations, the adaptive scheme automatically decreases the time step size, whereas for a smaller number of iterations the adaptive scheme increases it.

The obtained distributions of stresses and phase-field are shown in Figure 2. The

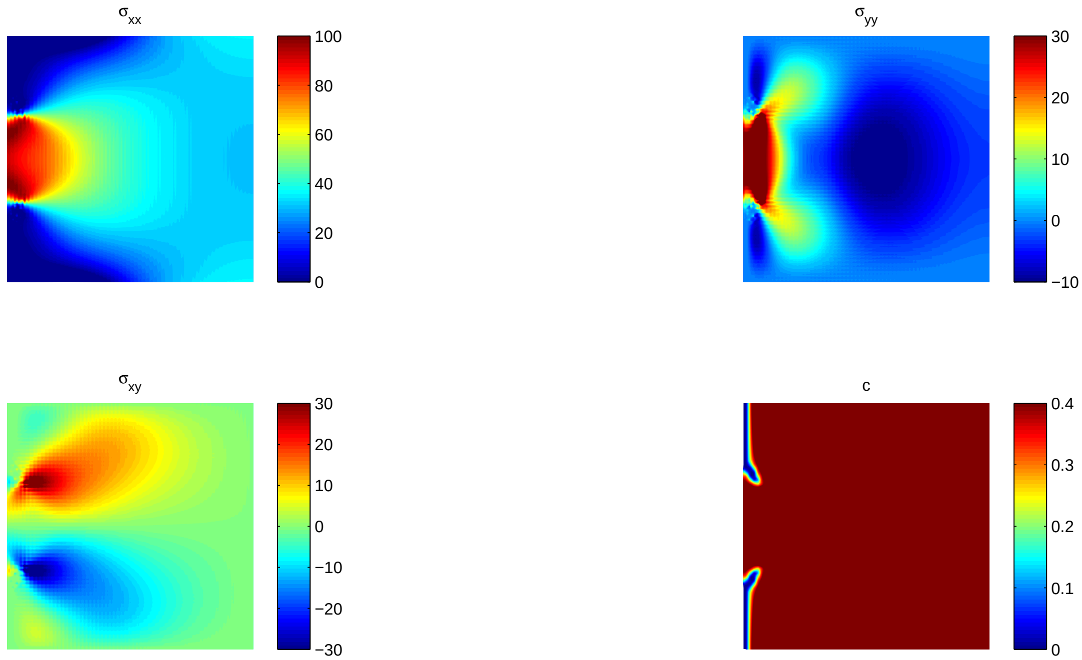

Figure 2: Square plate under quasi-static prescribed displacement: stresses and phase-field obtained on a 64×64 mesh for $\bar{u} = 4.5 \times 10^{-4}$

cracks initiate from the corners of the built-in edge at a displacement of $\bar{u} = 2.9 \times 10^{-4}$. At a displacement of $\bar{u} = 4.1 \times 10^{-4}$ the cracks start deviating from the vertical edge. As expected, high stress gradients can be observed at the crack tips.

The undeformed and deformed configurations of the plate are plotted in Figure 3.

5 CONCLUSIONS

- Discrete methods are well suited only for static fracture and a moderate number of cracks.

- Continuum damage methods are not effective when modelling large dominant cracks.

- The variational approach circumvents the implementation of complex crack-tracking algorithms and the need to describe the topology of the crack surface (the crack path is the natural outcome of the analysis).

- The numerical implementation of the variational approach requires a high resolution of the crack, which may lead to computationally expensive simulations.

Acknowledgements

The research leading to these results has received funding from the European Union Seventh Framework Programme (FP7/2007-2013) under grant agreement No. PIAPP-

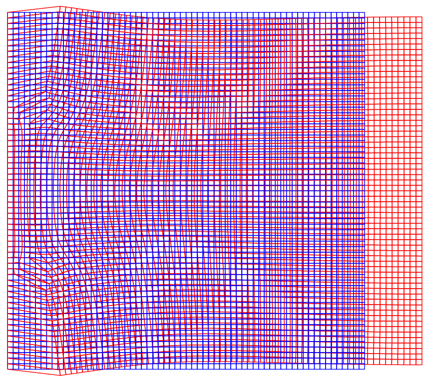

Figure 3: Square plate under quasi-static prescribed displacement: deformed (scaled) and undeformed configurations of the plate obtained for $\bar{u} = 4.5 \times 10^{-4}$

GA-284544-PARM-2.

REFERENCES

[1] Song, J.-H., Wang, H. and Belytschko, T. A comparative study on finite element methods for dynamic fracture. *Comput. Mech.* (2008) **42**:239–250.

[2] Miehe, C., Hofacker, M. and Welschinger, F. A phase field model for rate-independent crack propagation: robust algorithmic implementation based on operator splits. *Comput. Method Appl. M.* (2010) **199**:2765–2778.

[3] Borden, M.J., Verhoosel, C.V., Scott, M.A., Hughes, T.J.R. and Landis, C. M. A phase-field description of dynamic brittle fracture. *Comput. Method Appl. M.* (2012) **217-220**:77–95.

[4] Hakim, V. and Karma, A. Laws of crack motion and phase-field models of fracture. *J Mech. Phys. Solids* (2009) **57**:342–368.

[5] Bourdin B., Francfort, G.A. and Marigo, J.-J. Numerical experiments in revisited brittle fracture. *J Mech. Phys. Solids* (2000) **48**:797–826.