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# AN ISOGEOMETRIC REISSNER-MINDLIN SHELL WITH LAGRANGE BASIS

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**Abstract.** In this paper, we propose a new type of isogeometric Reissner-Mindlin degenerated shell for linear analysis. Degenerated shell is a frequently used shell type deduced by degeneration from 3D solid element, and has a very easy implementation procedure. When extended to isogeometric shell analysis, it suffers from the fiber vector determination problem. In this paper, we propose a method based on the use of lagrange basis. Due to the interpolative property of the lagrange basis, the fiber vectors can be naturally defined. This method keeps the geometric exact character of the isogeometric analysis and avoids the difficulties in the definition of the fiber vectors. At the same time, the rotation boundary conditions can be easily imposed. There will be different numbers of degrees of freedom for the displacements and the rotations in a single element. Examples show that the method proposed is simple and effective.

## 1 INTRODUCTION

Isogeometric analysis (IGA) is a numerical method introduced by Hughes et al. [1]. It has been successfully used in various fields where there traditional finite element analysis (FEM) is used [2, 3, 4]. The character of this method is specified by descripting the unknown field with the same basis used in the description of geometries, so it can also be viewed as a special isoparametric FEM with exact geometric mapping.

Isogeometric shell analysis is a combination of traditional shell models and the isogeometric approximation method. Kirchhoff-Love and Reissner-Mindlin shell theories are the two main shell models used in engineering. The former is normally referred as thin shell theory, for it doesn't account for shear deformation and can be used for thin shell model. The latter is shear deformable and can also be used for thick shell. There are also other shell types like solid shell or 3D shell. Book [5] gives a throughout introductions of these shell models, one is suggested to read it for a deep understanding.

Various IGA shell models have been developed. The basic idea of these formulation is that replacing the lagrange basis functions with the basis used in isogeometric analysis. Kirchhoff-Love shell model is rarely used in traditional FEM analysis, it is not due to its effectiveness but its complexity in implementation. It needs at less $C_1$ continuity basis functions, but the construction of $C_1$ lagrange basis functions is difficult. This severely limits its application in engineering. The emergence of IGA method casts light on this problem, since NURBS basis functions naturally provide high continuity basis functions. J.Kiendl [6] proposed the first IGA Kirchhoff-Love shell in 2009. Another IGA shell formulation based this model can be found in [7]. But the Kirchhoff-love assumptions are not exactly fulfilled, the normal vector used in this paper just hold in the quadrature points instead of the entire domain. Paper [8] presents an IGA shell based on Reissner-Mindlin shell model. The author uses degenerated method to formulate it. Paper [9] is also in this type with an improvement in the calculation and update of directors. It is deduced from continuum theory and is suitable for linear and non-linear analysis. Paper [10] can be viewed as a blended application of paper [7] and [8]. A hierarchic family of IGA shells is presented in paper [11]. Paper [12] and [13] present a type of solid shell.

In this paper, a new Reissner-Mindlin IGA shell for linear analysis is developed. It can avoid the complex procedures in the determination of fibers victors. In our method, NURBS basis are used to express the mid-surface and the Lagrange basis are used to express the fibers. The use of Lagrange basis allows a natural choice of fibers vectors. As a result, the movements of the mid-surface will be expressed with NURBS basis and the rotations of the fibers will with Lagrange basis. There will be different numbers of degrees of freedoms for displacements and rotations in a single element. The node's degrees of freedom concept disappears.

The paper is arranged as following: In section 2, the formulation of degenerated solid shell and its implementation with IGA and lagrange basis are stated. In section 3, three examples from shell obstacle courses are used to test this new IGA shell, various results are presented. In section 4, a brief conclusion is summarized.

## 2 FORMULATION

### 2.1 ISOGEOMETRIC ANALYSIS

Isogeometrix method uses the basis functions which are used in the description of geometries for analysis. It is introduced by TJ.Hughes in 2005 by using NURBS in analysis. Other basis functions such as T-Spline [15], PHT spline (Spline on Hierarchy T-mesh) [14] can also be used in IGA analysis. In this paper, we just use NURBS basis functions. Its definition is as following [16],

$$R_{i,j}^{p,q} = \frac{N_{i,p}\left(\xi\right) N_{j,q}\left(\eta\right) w_{i,j}}{\sum\limits_{i=1}^{n} \sum\limits_{j=1}^{m} N_{i,p} N_{j,q} w_{i,j}} \tag{1}$$

$N_{:,p}$ and $N_{:,q}$ are one dimensional B-spline basis which are defined by the knots vector $\{\xi_1, \xi_2, \cdots, \xi_{p+n}\}$ and $\{\eta_1, \eta_2, \cdots, \eta_{q+n}\}$. $p$ , $q$ are the degrees of basis in the two parameter directions respectively. $w$ is the weight dominated by the geometry shape description. The definition of B-spline basis is as following:

$$N_{i,1} = \begin{cases} 1 & if\ \xi_i \le \xi < \xi_{i+1} \\ 0 & otherwise \end{cases}$$

for $p > 1$ :

$$N_{i,p} = \begin{cases} \frac{\xi-\xi_i}{\xi_{i+p-1}-\xi_i} N_{i,p-1} + \frac{\xi_{i+p}-\xi}{\xi_{i+p}-\xi_{i+1}} N_{i,p-1} & if\ \xi_i \le \xi < \xi_{i+1} \\ 0 & otherwise \end{cases}$$

In IGA analysis, the above basis functions are determined by the geometries. Once the shape description is fixed, the roughest mesh for analysis is got, and then after knots insertion and order elevation procedures, a more fine mesh can be created without changing the geometries shape but adding more basis functions. This procedure is referred as the mesh refinement in IGA, see in [1].

The common used NURBS geometries in IGA are descripted by closed knots vectors which means that the first and end knot points should appear $p$ times. Each non-zero interval of the knots vector will be an element in IGA (in one dimension, in higher dimension it will be the tensor multiplication of the one dimension).

### 2.2 SHELL FORMULATION

Our method is based on degenerated shell [17, 18]. Degenerated shell is the mostly used method in shell analysis for its simplicity. The idea is to discretize the shell into solid element, and then use linear shape functions in the thickness direction. After simplifications, the geometry of the shell will be expressed will a parametric mid-surface and an interpolated fiber vectors in the thickness direction. Correspondingly, the kinemics of the geometry will be expressed by the translation of the nodes in the mid-surface and the rotations of the fiber vectors. The detailed procedures can be seen in [17] and [18].

The geometry expression of our shell model is as following:

$$\mathbf{x} = \sum_A N_A(\varepsilon, \eta)\,\mathbf{x}_A + \frac{h}{2}\varsigma \sum_B N_B(\varepsilon, \eta)\,\mathbf{y}_B \tag{2}$$

Figure 1 depicts the 3D shell expression. $N_A$ is the NURBS basis function. $N_B$ is the lagrange basis functions. $h$ is the thickness. $\mathbf{y}_B$ is the unite fiber vector in node B which indicates the thickness direction.

With the inextensibility of the fiber vectors and small deformation assumptions, the displacement of the shell can be expressed as following,

$$\mathbf{u} = \sum_A N_A(\varepsilon, \eta)\,\mathbf{u}_A + \frac{h}{2}\varsigma \sum_B \mathbf{w}_B \times \mathbf{y}_B$$

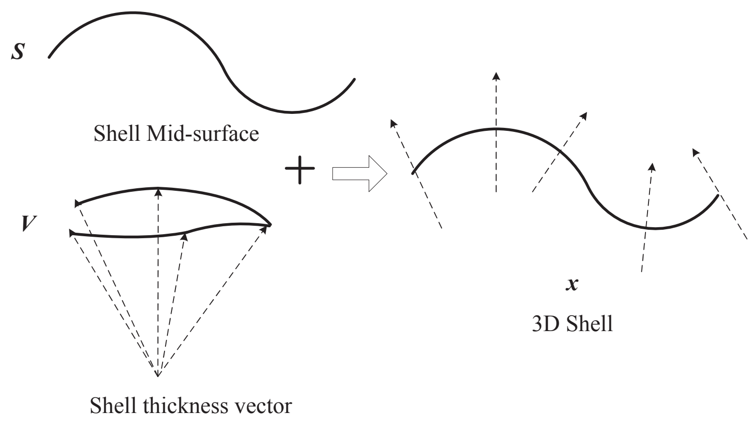


**Figure 1**: 3D shell geometry description

$\mathbf{w}_B$ is a vector comprised of three small items which describe the rotation angles along three global coordinate axis. If the vector $\mathbf{w}$ is expressed in a node coordinate system one of whose axis coincides with $\mathbf{y}_A$, such as $\mathbf{w}_B = \theta_{B1}\mathbf{e}_{B1} + \theta_{B2}\mathbf{e}_{B2} + \theta_{B3}\mathbf{e}_{B3}$, $\mathbf{e}_{B3}$ is equal to $\mathbf{y}_B$, it can be got that,

$$\mathbf{u} = \sum_A N_A(\varepsilon, \eta)\,\mathbf{u}_A + \frac{h}{2}\varsigma \sum_B N_B(\varepsilon, \eta)\left(-\theta_{B1}\mathbf{e}_{B2} + \theta_{B2}\mathbf{e}_{B1}\right) \tag{3}$$

Here, $\mathbf{u}_A$ is node displacement, it is attached to each control points. $\mathbf{e}_{B2}$ and $\mathbf{e}_{B1}$ are the rotation axis attached to each rotation node and $\theta_{B2}$ and $\theta_{B1}$ are the corresponding rotation angles.

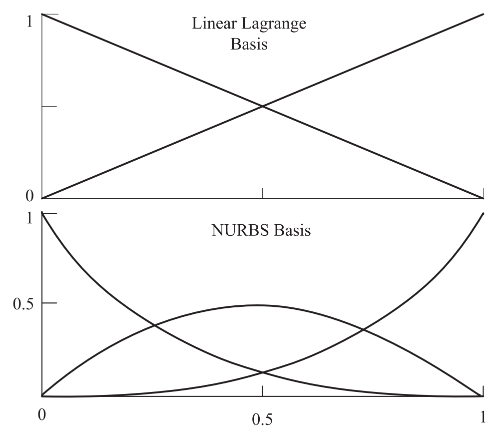


**Figure 2**: Linear lagrange and NURBS basis functions in an element interval [0 1].

It can be seen that in an NURBS IGA element, there exist NURBS basis functions and lagrange basis functions at the same time. In our method, the nodes for rotation are located in each corner of the NURBS element. Figure 2 depicts the one dimensional condition.

The linear strain tensor is used as following,

$$\mathbf{E} = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$$

The strain energy can be got as,

$$W_{\text{int}} = \frac{1}{2}\int_{\Omega} \mathbf{E} : \mathbf{C} : \mathbf{E} \, d\Omega \tag{4}$$

And then the stiffness matrix will be as following,

$$K_{rs} = \frac{\partial W_{\text{int}}}{\partial u_r \partial u_s} \tag{5}$$

The tensor in equation (4) needs to be expanded in a coordinate system to calculate the integrant. In another aspect, in order to modify the material tensor to reach the zero normal stress condition, the material tensor will also be expanded in a so-called lamina coordinate system. The construction of this coordinate system can be referred to [17].

## 3 EXAMPLES

Three benchmarks from shell obstacle courses are used to test our method. They are scordelis-lo roof problems, pinched cylinder problem and hemisphere problem [19]. For each problem, the degrees of the NURBS used are $\{3, 5, 7, 9\}$ . The meshes are uniformly divided with the segments number $\{5, 10, 15, 20, 30, 40\}$ .

### 3.1 SCORDELIS-LO ROOF

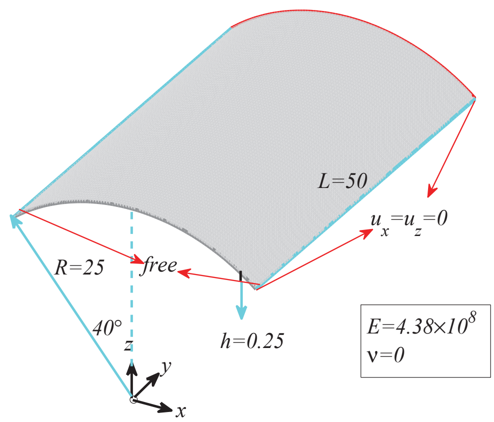


**Figure 3**: Scordelis-Lo roof problem setup.

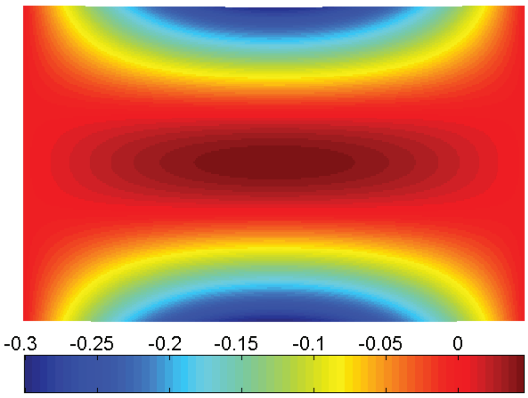


**Figure 4**: Scordelis-Lo roof, $u_z$ contour.

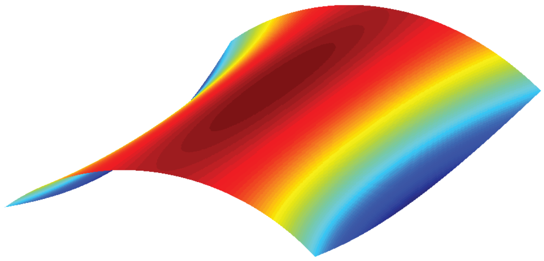

**Figure 5**: Scordelis-Lo roof, deformed shape.

Problem setup is depicted in Figure 3. The structure is applied a gravity of 90. Figure 6 depicts the $u_z$ displacement varies with $h$ refinements. The converged value is $-0.3009$ with $40 \times 40$ elements. It is comparable with that reported in paper [6]. Figure 4 depicts the $u_z$ contour plot. Figure 5 depicts the deformed structure with a scaling factor 10.

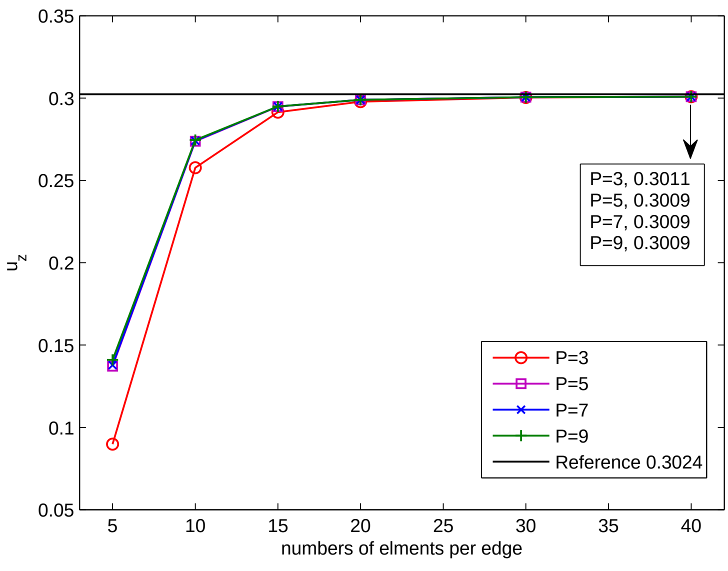


**Figure 6**: Scordelis Lo roof, $u_z$ varies with $h$ refinement.

## 3.2 PINCHED CYLINDER

Figure 7 depicts the cylinder shell problem setup. Due to the symmetry of the problem, one quarter of the structure is modeled and depicted. The $u_z$ displacement with $h$

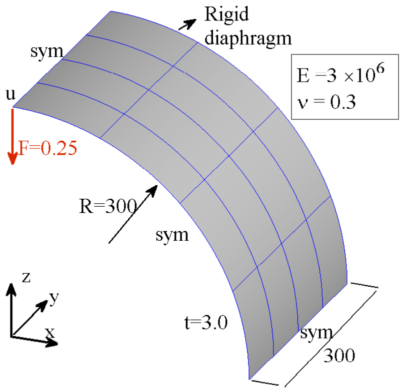


**Figure 7**: Pinched cylinder problem setup.

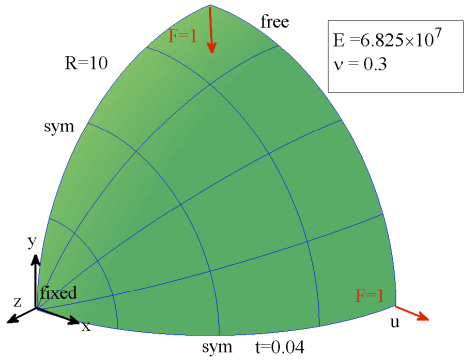


**Figure 8**: Hemisphere problem setup.

refinements is depicted in Figure 9. The converged value is comparable to the reference value.

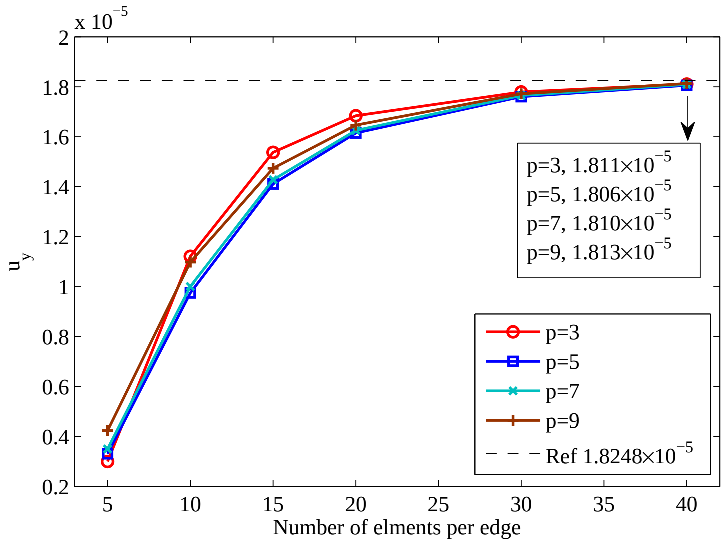


**Figure 9**: Pinched Cylinder, $u_z$ varies with $h$ refinement.

### 3.3 HEMISPHERE SHELL

Figure 8 depicts the hemisphere problem setup. Due to the symmetry, one quarter of the structure is modeled. Figure 10 depicts the $u_y$ displacement with $h$ refinement. It can be seen that the results got are comparable to that has been reported.

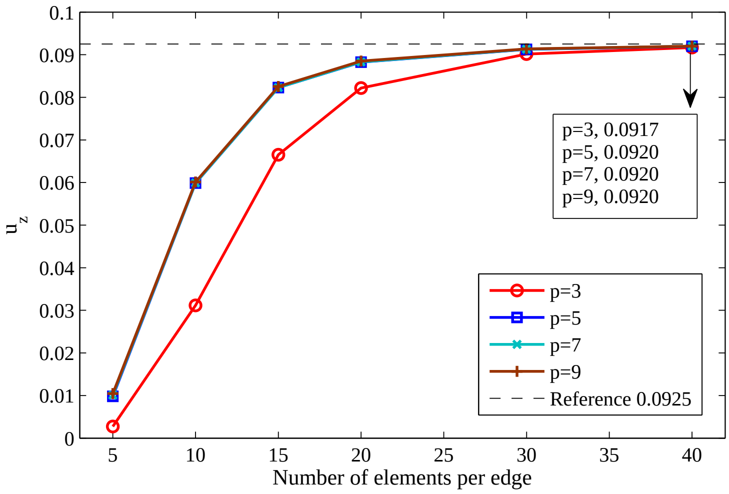


**Figure 10**: Hemisphere, $u_y$ varies with $h$ refinement.

## 4 CONCLUSIONS

In this paper, we propose a new type of isogeometric reissner mindlin degenerated shell for linear analysis. By introducing lagrange basis functions into isogeometric Reissner-Mindlin shell formulation. The definition of fiber vectors problem is successfully avoided. This strategy not only keeps the exactly expression of the shell shape but also allows an easier imposition of rotation boundary conditions due to the interpolative character of the Lagrange basis. It should be noticed that the expressions of mid-surface and directors are independent, there will be different numbers of degrees of freedom for the expression of the translation of the mid-surface and the rotations of the directors in an element.

Further research about this topic is undergoing, new results may come out when this conference is held.

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