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ADAPTIVE GALERKIN METHOD WITH RELEVANT BASIS FUNCTIONS FOR PDES WITH BOUNDARY CONDITIONS

BING LI[*], LUOFENG HAN[*] AND SHUANGLU QUAN[*]

[*] State Key Laboratory for Manufacturing Systems Engineering,
Xi'an Jiaotong University, 710049, Xi'an, P. R. China
e-mail: bli@mail.xjtu.edu.cn, hanluofeng.2008@163.com

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Abstract. As a useful tool for solving partial differential equations, Galerkin method has been developed for solving different problems through constructing different types of basis functions. Previous construction methods mainly focused on constructing common and optimal basis functions, neglecting the effect of the known information existing in differential equation itself. To adequately utilize the existing information, relevant basis function (RBF) based on optimal thought is proposed in this paper. The concept of relevant basis function is defined, and its properties, including similarity, adaptability and optimality, are described. Different from traditional basis functions, RBFs are formed by two parts. Ones are the known relevant basis functions, constructed by utilizing the known conditions reflected by the form of differential equation, and the others are the unknown relevant basis functions with the known form determined by the known conditions, including undetermined part. After constructing relevant basis functions, the adaptive Galerkin method with relevant basis functions is designed for solving partial differential equations with boundary conditions, mainly including two aspects. One is that the coefficients of basis functions are obtained by Galerkin method, and the other is that the undetermined part of unknown relevant basis functions is solved adaptively by iterative method. Numerical examples demonstrate that the adaptive Galerkin method with relevant basis functions is flexible and accurate with economical algorithm for solving partial differential equations with boundary conditions.

1 INTRODUCTION

In recent years, Galerkin methods have been developed as a powerful tool for mathematical analysis and engineering computation, especially for solving integral and differential equations. The important step of Galerkin method is to construct basis functions or shape functions to approximate the objective solution. If selecting appropriate basis functions, the solving results are approximate to the exact solution with economical algorithm. In other words, constructing proper basis functions can improve the accuracy and practicality of Galerkin method for solving partial differential equations. Different types of basis functions have been constructed to solve different problems as follows.

Orthogonal polynomials are often used as basis functions, with the advantages of spectral convergence and the satisfaction of boundary conditions. On the non periodic canonical interval $[-1, 1]$, the Jacobi polynomials are a well-known class of polynomials exhibiting spectral convergence [1,2], of which particular examples are Chebyshev polynomials and Legendre polynomials [3,4]. Chebyshev polynomials are often a popular choice since, via their links with Fourier methods, they have a fast transform [5,6]. Other polynomials were also used as basis functions, e.g., Yousefi, Barikbin, et al. [7] implemented the Ritz-Galerkin method in Bernstein polynomial basis to give approximate solution of a parabolic partial differential equation with non-local boundary conditions.

Other basis functions were also constructed to solve special engineering problems. Sauter and Veit [8] applied Galerkin method in space and time in order to solve the three-dimensional wave equation in unbounded domains, with smooth and compactly supported temporal basis functions. Dag, Canivar et al. [9] provided the numerical solutions of the time-dependent advection-diffusion problem by using B-spline finite element methods with quadratic and cubic B-spline basis functions. Dehghana, Yousefib et al. [10] presented the properties of Bernstein multi-scaling functions to reduce the main problem to the solution of nonlinear algebraic equations, and the B-spline scaling functions were used in the Ritz–Galerkin technique to solve the inverse problem.

Although various basis functions have been constructed creatively to successfully extend the concept and content of basis function or shape function, there are still two disadvantages existing. One is that the construction of basis functions is not enough simple and economical, and the other is that the known conditions are not adequately utilized in the construction of basis functions.

In this paper, to make use of the known conditions of differential equation, we propose the concept of relevant basis functions based on optimal thought. The most important property of relevant basis function is similarity that relevant basis function is similar to one item of the objective solution of differential equation. Relevant basis functions are divided into two parts. Ones are the known relevant basis functions, constructed by the known information of differential equation, and the others are the unknown relevant basis functions with the known form determined by the existing information, including the undetermined part solved by iterative method. Based on relevant basis functions, the adaptive Galerkin method is designed for solving partial differential equations with boundary conditions. Numerical examples demonstrate that the adaptive Galerkin method with relevant basis functions is an accurate and effective tool for the solution of PDEs with boundary conditions.

2 RELEVANT BASIS FUNCTIONS

We have noticed that that the existing information of differential equation is useful for constructing basis functions, thus in this section, relevant basis function based on known conditions is presented to make up the drawbacks of traditional basis functions. The concept of relevant basis function is defined, and its properties, including similarity, adaptability and optimality, are described.

2.1 Definition of relevant basis function

The exact solution of partial differential equation includes different types of items, and each item is a certain function. If selecting these items as basis functions, we will get the exact solution. The fact is that these items are impossible to be obtained, but the approximation can be made to construct appropriate basis functions. When selecting basis functions, if some basis function is similar to one item of the exact solution, this basis function is called as relevant basis function (RBF). In other words, RBF is the relevant function of the exact solution of differential equation.

2.2 Properties of relevant basis function

Relevant basis function has three basic characteristics, including similarity, adaptability and optimality. These three properties are described in detail as follows.

(1) Similarity

Relevant basis function is similar to one item of the exact solution of differential equation. However, the exact solution is undetermined, and thus it is impossible to get relevant basis functions from the exact solution. It is well-known that differential equation itself reflects some information of the exact solution, and thus relevant basis functions can be derived by utilizing the existing information of differential equation. For example, $\sin(t)$ is one item of a differential equation, we can be certain that the exact solution includes the items relevant to the trigonometric function $\sin(t)$. Based on known conditions, then $\sin(t)$, $\cos(t)$, $\sin(\theta(t))$ and $\cos(\theta(t))$ can be selected as relevant basis functions. $\theta(t)$ is undetermined in $\sin(\theta(t))$ and $\cos(\theta(t))$.

(2) Adaptability

Relevant basis function can be known function or unknown function. The known basis function could be selected from the family of relevant functions based on the known conditions of differential equation. The form of unknown relevant basis function is determined by the form of differential equation, but some part in detail is undetermined, which can be obtained by iterative methods. Here adaptability lies in two respects. One is that the selection of basis functions is based on the known conditions of differential equation, and the other is that some relevant basis functions include the unknown part solved adaptively by iterative algorithm. The approximate solution of differential equation can be expressed as

$$\tilde{u}(t) = \sum_{j=1}^{M_1} a_j \phi_j(t) + \sum_{k=1}^{M_2} b_k \sigma_k(t) \tag{1}$$

where a_j ($j = 1,2,\cdots,M_1$) and b_k ($k = 1,2,\cdots,M_2$)represent the coefficients of basis functions, ϕ_j ($j = 1,2,\cdots,M_1$)are the known relevant basis functions, and σ_k ($k = 1,2,\cdots,M_2$) are the unknown relevant basis functions.

(3) Optimality

If the selected basis functions can approximate to the items of the exact solution of differential equation, we would get the accurate approximation to the objective solution with small computational complexity. Due to the first and second properties of relevant basis function, it is proper to select relevant basis functions to constitute the approximate solution. Compared with other basis functions, the number of relevant basis functions used to approximate to the unknown function is fewer for the same degree of accuracy. For a single unknown function $u(t)$, it follows that

$$u(t) \approx \sum_{j=1}^{P} N_j u_j = \mathbf{Nu}\,, \quad N_j \in D \tag{2}$$

where $\mathbf{N} = [N_1, N_2, \cdots, N_P]$ is a set of basis functions, P is the number of the basis functions, $\mathbf{u} = [u_1, u_2, \cdots, u_P]$ represents the coefficient set of the basis functions, and D represents the function library. If D is the set of relevant basis functions, P takes the minimum value.

2.3 Approximate solution of PDEs by using RBFs

The general form of one-dimensional partial differential equation can be written as

$$\begin{cases} A(u) = u''(t) + pu'(t) + qu(t) + \sum_{i=1}^{N_1} \alpha_i \cos(\gamma_i(t)) + \sum_{k=0}^{N_2} \beta_k t^k = 0 \\ B(u) = 0 \end{cases} \tag{3}$$

where p and q are constant, α_i ($i = 1,2,\cdots,N_1$) and β_k ($k = 0,1,\cdots,N_2$) represent the known coefficients.

For the differential equation (3), $\cos(\gamma_i(t))$, $\sin(\gamma_i(t))$ ($i = 1,2,\cdots,N_1$) and t^k ($k = 0,1,\cdots,N_2$) could be selected as the known relevant basis functions. The unknown relevant basis functions can not be polynomials, for that t^k ($k = 0,1,\cdots,N_2$) have included all the relevant polynomials. So the hiding unknown basis functions are assumed as $\sin(\theta(t))$ and $\cos(\theta(t))$ based on the form of the differential equation (3), and then the approximation to the unknown function $u(t)$ could be expressed as

$$\tilde{u}(t) = \sum_{j=1}^{Q} a_j \phi_j(t) + b_1 \cos(\theta(t)) + b_2 \sin(\theta(t)) \tag{4}$$

where a_j ($j = 1,2,\cdots,Q$), b_1 and b_2 are the unknown coefficients, and $\phi_j(t)$ ($j = 1,2,\cdots,Q$) represent the known relevant basis functions, including $\cos(\gamma_i(t))$, $\sin(\gamma_i(t))$ ($i = 1,2,\cdots,N_1$) and t^k ($k = 0,1,\cdots,N_2$).

3 ADAPTIVE GALERKIN ALGORITHM WITH RELEVANT BASIS FUNCTIONS

Based on relevant basis functions, the adaptive Galerkin method is designed for solving partial differential equations with boundary conditions. In the adaptive Galerkin algorithm, two aspects should be guaranteed to obtain the accurate approximation to the objective solution. One is to adequately utilize the existing information of differential equation, and the other is to select proper iterative method with good convergence for $\theta(t)$. The first condition is satisfied by utilizing all the relevant functions reflected by the form of differential equation, and for the second condition, Aitken iterative method with good convergence is selected to obtain the accurate approximation of $\theta(t)$. Taking the differential equation (3) for example, the adaptive Galerkin algorithm is shown as follows.

Step 1: Select relevant basis functions and write out the approximate solution form (4).

Step 2: Set the initial value of the unknown part $\theta(t)$, and make sure that the initial value approximates to the true value.

Step 3: Write out the equivalent integral form of the differential equation (3)

$$\int_{\Omega}\mathbf{v}^{\mathrm{T}}A(\widetilde{u})d\Omega+\int_{\Gamma}\overline{\mathbf{v}}^{\mathrm{T}}B(\widetilde{u})d\Gamma=0 \tag{5}$$

where $\mathbf{v}$ and $\overline{\mathbf{v}}$ are arbitrary functions, and apply Galerkin method to obtain the coefficients of basis functions as follow.

$$\int_{\Omega}\mathbf{N}^{\mathrm{T}}A(\widetilde{u})d\Omega-\int_{\Gamma}\mathbf{N}^{\mathrm{T}}B(\widetilde{u})d\Gamma=0 \tag{6}$$

where $\mathbf{N}$ is basis function set, including $\cos(\gamma_i(t))$, $\sin(\gamma_i(t))$ ($i=1,2,\cdots,N_1$) and t^k ($k=0,1,\cdots,N_2$).

Step 4: Update $\widetilde{u}(t)$ with the solved coefficients, and bring $\widetilde{u}(t)$ into the differential equation (3) to obtain the residual equation $R(\theta(t),t)=0$.

Step 5: Separate $\theta(t)$ into the discrete points $\theta_1,\theta_2,\cdots,\theta_M$ in the domain Ω. Every point (θ_i,t_i) ($i=1,2,\cdots,M$) satisfies the equation $R(\theta(t),t)=0$, thereby the following equations are obtained.

$$\begin{cases} R(\theta_1,t_1)=0 \\ \quad\vdots \\ R(\theta_M,t_M)=0 \end{cases} \tag{7}$$

Step 6: Obtain the iterative form $\theta_i^k=\varphi(\theta_i^{k-1},t_i)$ ($i=1,2,\cdots,M$) from Eq. (7), and use Aitken iterative algorithm to iterate θ_i.

Step 7: If $\left\|\theta_i^k-\theta_i^{k-1}\right\|_\infty\le\xi$ (or $\left\|R(\theta_i^k,t_i)\right\|_\infty\le\xi$), stop. Otherwise, polynomial interpolation for θ_i^k ($i=1,2,\cdots,M$) is carried out to obtain $\theta(t)$, that is

$$\theta(t)=\text{interpolate}\,(\theta_i^k,t_i),\ i=1,2,\cdots,M \tag{8}$$

and then go to step 3.

The flow chart of the adaptive Galerkin method is shown in Figure 1.

4 NUMERICAL EXAMPLES

We present numerical experiments to demonstrate the efficiency and accuracy of the adaptive Galerkin method with relevant basis functions. In this section, two one-dimensional differential equations with boundary conditions are solved by the adaptive Galerkin method, and comparison is carried out between the solving results and the exact solution of differential equation.

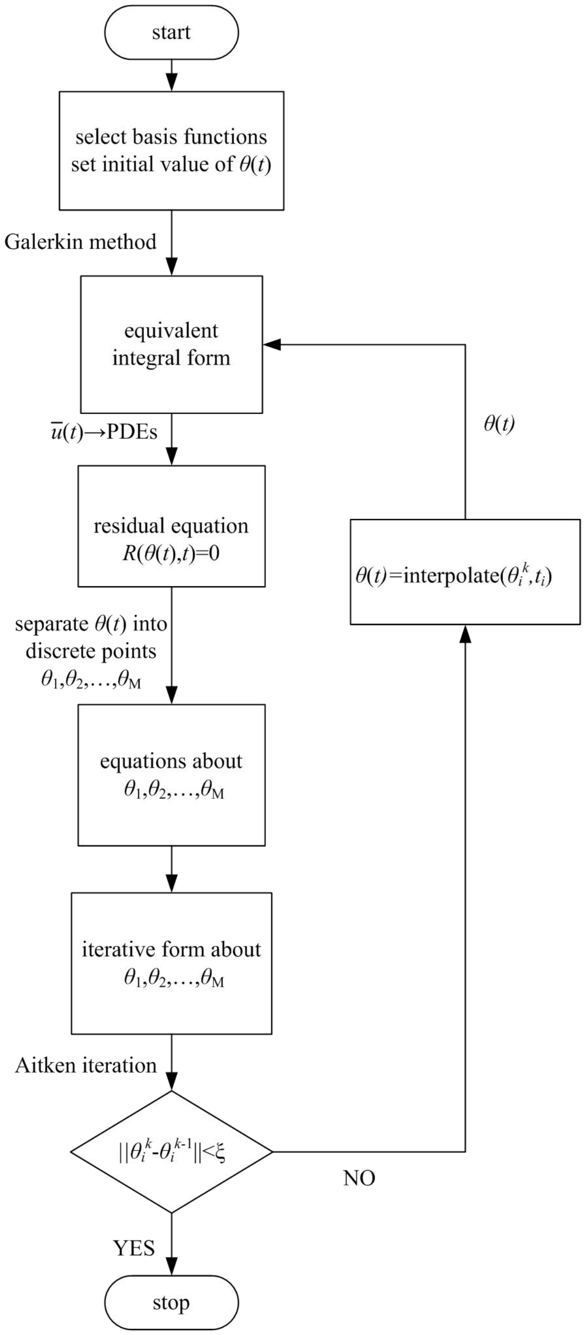

Figure 1: Flow chart of the adaptive Galerkin method

Example1 Consider a PDE with boundary condition as follow

$$u''(t)+u(t)=f\ ,t\in[0,1]$$

with the boundary condition $u\big|_{t=0,1}=0$; the right-hand function is $f=-t$ and the exact solution is $u(t)=\sin(t)/\sin(1)-t$.

For the above differential equation, t is the known information, and thus t^1 and t^0 can be selected as the known relevant basis functions based on the properties of relevant basis function. At the same time, the unknown relevant basis functions could be assumed as $\sin(\theta(t))$ and $\cos(\theta(t))$ based on the form of the differential equation, and thus the approximate solution can be written as $\tilde{u}(t) = a_1 t^1 + a_2 t^0 + b_1 \sin(\theta(t)) + b_2 \cos(\theta(t))$. In the approximate form, $\theta(t)$ is the undetermined part of the unknown relevant basis functions, and it could be obtained by iterative algorithm.

The initial value of $\theta(t)$ is set as $2t$, and then the coefficients of basis functions, a_i and b_i ($i = 1,2$), can be obtained by Galerkin method through solving the integral equations of the differential equation and boundary conditions. Then update the approximate solution $\tilde{u}(t)$ by using the solved coefficients a_i and b_i ($i = 1,2$), and bring $\tilde{u}(t)$ into the differential equation to obtain the residual equation $R(\theta(t), t) = 0$. Furthermore, separate $\theta(t)$ into eleven discrete points, $\theta_1, \theta_2, \cdots, \theta_{11}$, in the domain $[0,1]$. Because every point (θ_i, t_i) ($i = 1,2,\cdots,11$) satisfies the equation $R(\theta(t), t) = 0$, we get a set of nonlinear equations about $\theta_1, \theta_2, \cdots, \theta_{11}$, which can be solved by Aitken iterative method. Finally, the termination threshold ξ is set as 10^{-4}, and if the results are not satisfied the termination condition, polynomial interpolation (the highest power of polynomial is selected as 1.) for θ_i^k ($i = 1,2,\cdots,11$) is carried out to obtain $\theta(t)$.

The iterative results are shown in the Figure 2 (-o-, 1st iterative results; -□-, 3rd iterative results; -◊-, 7th iterative results; —, exact solution.). As shown in Figure 2, we can obtain that the iterative results approach the exact solution by degrees with the iterative times. The curve of the 7th iterative results is basically in coincidence with that of the exact solution. Figure 3 displays the errors of the 7th iteration, showing that the errors of the 7th iteration are within 0.001% in the domain [0, 1].

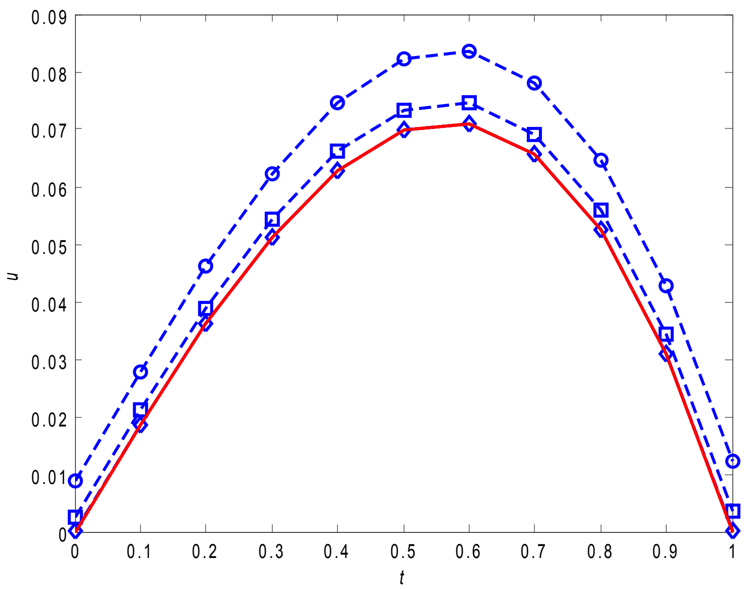

Figure 2: Iterative results of the differential equation. -o-, 1st iterative results; -□-, 3rd iterative results; -◊-, 7th iterative results; —, exact solution

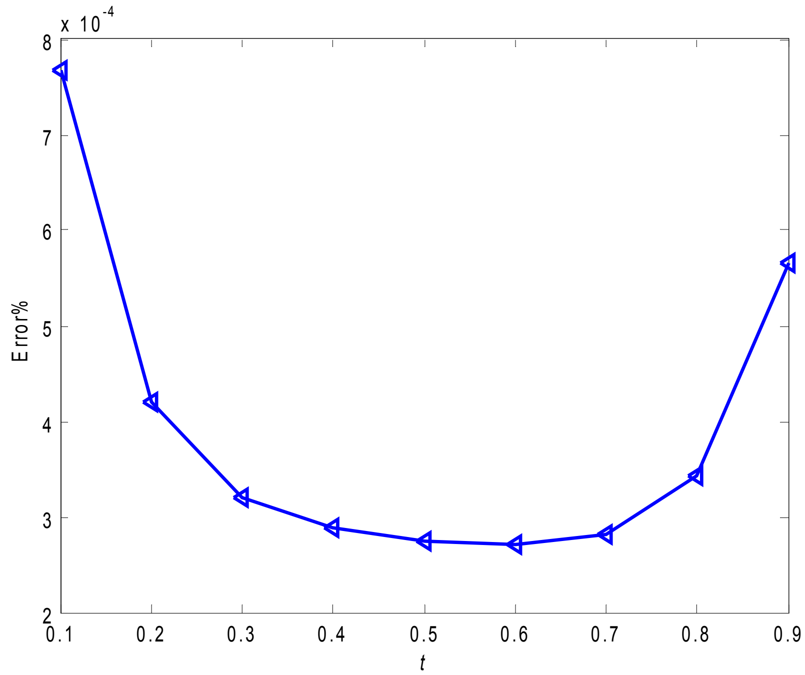

Figure 3: Errors of the 7th iteration

Example 2 Consider another PDE with boundary conditions as follow

$$u''(t)+4u(t)=f\ ,t\in[0,1]$$

with the boundary conditions $u\big|_{t=0}=1.6667, u\big|_{t=1}=0.5958$; the right-hand function is $f=10\sin(t)$ and the exact solution is $u(t)=5\cos(2t)/3-5\sin(2t)/3+10\sin(t)/3$.

Simultaneously, for the above differential equation, $\sin(t)$ is the known condition, and thus $\sin(t)$ and $\cos(t)$ can be selected as the known relevant basis functions. The unknown relevant basis functions are assumed as $\sin(\theta(t))$ and $\cos(\theta(t))$ based on the form of the above differential equation, and thus the approximation can be written as $\tilde{u}(t)=a_1\sin(t)+a_2\cos(t)+b_1\sin(\theta(t))+b_2\cos(\theta(t))$. In the approximate form, $\theta(t)$ is the undetermined part of the unknown relevant basis functions, and it could be obtained by iterative algorithm. The initial value of $\theta(t)$ is set as $3t$, and the termination threshold ξ is set as 5×10^{-3} .

The iterative results of the differential equation are shown in the Figure 4 (-o-, 1st iterative results; -□-, 3rd iterative results; -◊-, 6th iterative results; —, exact solution.). As shown in Figure 4, we can obtain that the iterative results approach the exact solution by degrees with the iterative times. The curve of the 6th iterative results is basically in coincidence with that of the exact solution. Figure 5 displays the errors of the 6th iteration, showing that the errors of the 6th iteration are within 0.016% in the domain [0, 1].

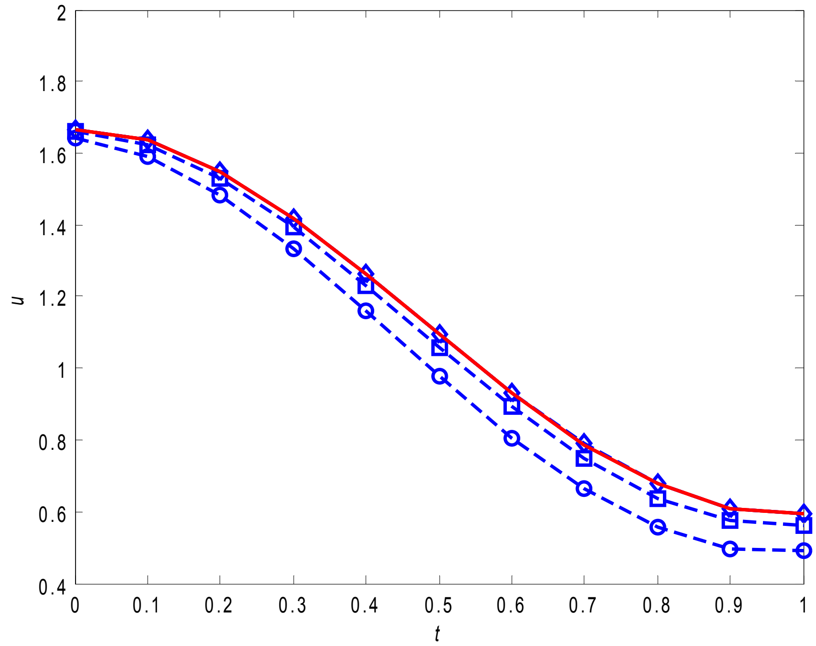

Figure 4: Iterative results of the differential equation. -o-, 1st iterative results; -□-, 3rd iterative results; -◊-, 6th iterative results; —, exact solution

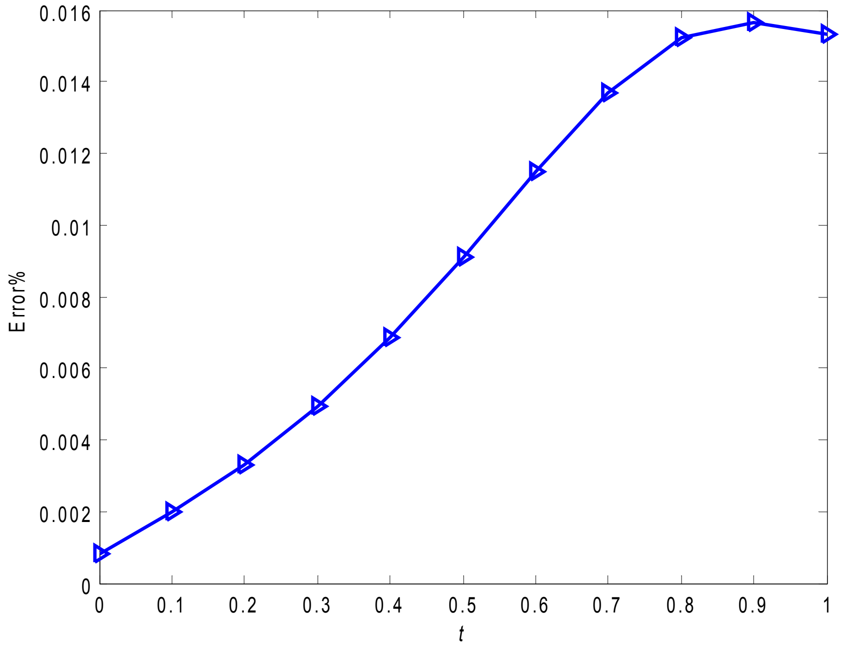

Figure 5: Errors of the 6th iteration

From the above two examples, we can see that the solving results by the adaptive Galerkin method can be accurately obtained with several iterative times. Furthermore, the construction of relevant basis functions is not complex by utilizing the existing information of differential equation, and the algorithm could be realized with good convergence. To sum up, the adaptive Galerkin method in this paper is suitable for solving ordinary differential equations with boundary conditions.

5 CONCLUSION AND DISCUSSION

To make up the drawbacks of traditional basis functions, relevant basis function is presented, utilizing the existing information of differential equation itself. Relevant basis functions are constructed based on the similarity principle, thereby constituting accurate approximation to the exact solution. Based on relevant basis functions, the adaptive Galerkin method is designed for solving one-dimensional partial differential equations with boundary conditions. The advantages of the adaptive Galerkin method with relevant basis functions lie in the following three aspects. Firstly, the existing information is adequately utilized in the construction of basis functions. Secondly, the designed algorithm is not complex with several iterations for solving equations. Thirdly, the solving results by this method are accurate with small errors.

We have noticed that the solving accuracy is sensitive to initial value, but this condition could be improved by improving algorithm convergence. At the same time, there is a possibility to construct more general form for solving complex problems in the future. Further work is continuing to demonstrate more achievements of relevant basis function for solving different types of differential equations.

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REFERENCES

[1] Livermore, P.W. Galerkin orthogonal polynomials. *J. Comput. Phys*. (2010) 229:2046-2060.

[2] Doha, E.H. Bhrawy, A.H. and Hafez, R.M. A Jacobi–Jacobi dual-Petrov–Galerkin method for third-and fifth-order differential equations. *Math. Comput. Model.* (2011) 53:1820-1832.

[3] Panigrahi, B.L. and Nelakanti, G. Legendre Galerkin method for weakly singular Fredholm integral equations and the corresponding eigenvalue problem. *J. Appl. Math. Comput*. (2013) 1-23.

[4] Chen, Y. Huang, F. Yi, N. et al. A Legendre-Galerkin spectral method for optimal control problems governed by Stokes equations. *SIAM J. Numer. Anal*. (2011) 49:1625-1648.

[5] Doha, E.H. Abd-Elhameed, W.M. and Bassuony, M.A. New algorithms for solving high even-order differential equations using third and fourth Chebyshev-Galerkin methods. *J. Comput. Phys*. (2012).

[6] Wang, Y.S. and Chien, C.S. A spectral-Galerkin continuation method using Chebyshev polynomials for the numerical solutions of the Gross-Pitaevskii equation. *J. Comput. Appl. Math*.(2011) 235:2740-2757.
[7] Yousefi, S.A. Barikbin, Z. and Dehghan, M. Ritz-Galerkin method with Bernstein polynomial basis for finding the product solution form of heat equation with non-classic boundary conditions. *Int. J. Nume. Methods Heat Fluid Flow*. (2012) 22:39-48.
[8] Sauter, S. and Veit, A. A Galerkin method for retarded boundary integral equations with smooth and compactly supported temporal basis functions. *Numer. Math*. (2013) 123:145-176.
[9] Dag, I. Canivar, A. and Sahin, A. Taylor-Galerkin method for advection-diffusion equation. *Kybernetes*. (2011) 40:762-777.
[10] Dehghan, M. Yousefi, S.A. and Rashedi, K. Ritz–Galerkin method for solving an inverse heat conduction problem with a nonlinear source term via Bernstein multi-scaling functions and cubic B-spline functions. *Inverse Probl. Sci. Eng*. (2013) 21:500-523.