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Nonlinear stability with the energy method: shallow water models and beyond

***Tomas Lundquist[1], Andrew R. Winters[2] and Jan Nordström[3]**

[1] Linköping University, SE-581 83 Sweden, tomas.lundquist@liu.se
[1] Linköping University, SE-581 83 Sweden, andrew.ross.winters@liu.se
[1] Linköping University, SE-581 83 Sweden, jan.nordstrom@liu.se

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Nonlinear stability has received increasing attention in recent years as a pertinent object of research in the analysis and numerical solution of fluid dynamics models. As opposed to the traditional method of linearization followed by an energy-based analysis, recent advances in nonlinear methodology have mainly focused on the notion of mathematical entropy. Roughly speaking, entropy-based approaches aim for the conservation or dissipation of some physically relevant convex function of the solution, thus avoiding non-physical blowups in highly non-linear environments such as under-resolved turbulence or shock propagation. On the continuous level, entropy bounds for conservation laws can, by definition, be obtained using a combination of both the chain rule and integration-by-parts. To obtain a similar result in the discrete analysis typically requires highly specialized flux formulations, which can be complicated to derive as well expensive to compute.

The energy method on the other hand aims specifically for a bound in mathematical energy, which may succinctly be defined as the L2 norm of some linearly independent set of solution variables. Or to put it in another way, it leads to a semi-bounded property of the spatial operator. For constant or variable coefficient problems, the concept of semi-boundedness can be utilized in proofs of well-posedness, while for numerical solutions using the method-of-lines, it instead leads to a generalization of the classical eigenvalue-based stability theory of ordinary differential equations. In addition, once the continuous equations are posed in the correct (i.e. semi-bounded) form, the energy method only requires the application of integration-by-parts. Continuous energy bounds (whether linear or nonlinear) can thus be mimicked discretely in a straightforward way, and with a limited computational overhead, by using any numerical method consistent with discrete summation-by-parts. Without the need to satisfy any further, problem dependent constraints, this leads to a simple and easy-to-use framework of discretization valid for general nonlinear models with a bounded energy

Thus, it can be argued, there are both theoretical and practical advantages associated with the energy method which are not present in an entropy-based approach. However, until recently only a few nonlinear equations, notably including the incompressible Navier-Stokes equations, have successfully been analyzed with the energy method without first linearizing and freezing the coefficients. The key to success lies in formulating the equations on a split form using a combination of skew-symmetric and symmetric, negative semi-definite terms. In this work we discuss and explore both the wider applicability and the advantages of an energy-based approach to nonlinear stability in fluid dynamics. This includes theoretical, practical and implementation related aspects. We exemplify our findings by considering split formulations of the nonlinear shallow water equations, both with and without including a bottom topography model for sediment transport.