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Unfitted mixed finite element methods

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Geometrically unfitted finite element methods such as CutFEM, Finite Cell, XFEM or unfitted DG methods have been developed and applied successfully in the last decades to a variety of problems ranging from scalar PDEs on stationary domains to systems of PDEs on moving domains and PDEs on level set surfaces. Two key techniques that enabled the success of these methods for a broad range of applications are

- the weak imposition of boundary conditions, e.g. via Nitsche method and
- stabilization techniques to deal with bad cuts such as ghost penalties or aggregations

These approaches combined with established tools of finite element methods allowed to apply and analyze unfitted methods in many fields. An important and powerful class of finite elements are mixed methods based on special vectorial finite elements such as $H(\text{div})$-conforming spaces. These are typically tailored to preserve conservation properties like mass conservation exactly in the discretization. A basic example is the mixed formulation of the Poisson problem which in the fitted case takes the following form:

Strong form:
Find σ, u with $u = 0$ on $\partial\Omega$, s.t.

$$\sigma - \nabla u = 0 \quad \text{in } \Omega, \qquad \text{(F)}$$
$$\operatorname{div} \sigma = -f \quad \text{in } \Omega, \qquad \text{(C)}$$

Mixed FEM:
Find $\sigma_h \in \Sigma_h \subset H(\text{div}, \Omega)$, $u_h \in Q_h \subset L^2(\Omega)$, s.t.

$$(\sigma_h, \tau_h)_\Omega + (\operatorname{div} \tau_h, u_h)_\Omega = 0 \text{ for all } \tau_h \in \Sigma_h, \quad (F_h)$$
$$(\operatorname{div} \sigma_h, v_h)_\Omega = (-f, v_h)_\Omega \text{ for all } v_h \in Q_h. \quad (C_h)$$

When looking for unfitted versions of these type of discretizations a major difficulty is stability, especially stability in the sense of the Ladyzhenskaya–Babuška–Brezzi (LBB) condition. An LBB condition that may be valid on fitted meshes for a given pair of spaces Σ_h and Q_h w.r.t. the background mesh can easily degenerate on arbitrary cuts. Additional stabilizations like the ghost penalty method which introduces couplings between u_h and v_h have been used in the literature to arrive at stable unfitted mixed discretizations. However, these stabilizations perturb the conservation property. In this talk we introduce a new approach that yields an LBB-stable discretization of the unfitted mixed problem without the need for stabilizations of ghost penalty type.