Load Capacity Ratios of Perfectly Plastic Bodies and Structures

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ABSTRACT

Consider a homogeneous isotropic elastic-perfectly plastic body $\Omega$ modeled by a bounded open subset of $\mathbb{R}^3$ having a Lipschitz boundary $\Gamma = \partial \Omega$. The body is assumed to be supported on an open subset $\Gamma_0$ of its boundary and $t$ denotes an external surface traction acting on the complementary part, $\Gamma_t$, of the boundary. Body forces may be included in the analysis but for the sake of simplicity we omit them here. We prove (see [1]) that there exists a maximal positive number $C$, to which we refer as the load capacity ratio, such that the body will not collapse plastically under any external traction field $t$ bounded by $Y_0C$, where $Y_0$ is the yield stress. Thus, while the limit analysis factor of the theory of plasticity pertains to a specific distribution of external loading, the load capacity ratio is independent of the distribution of the external loading and implies that no collapse will occur for any field $t$ on $\partial \Omega$ as long as $\operatorname{ess \ sup}_{x \in \partial \Omega} |t(y)| < Y_0C$. Collapse will occur for some $t$ for which the essential supremum of its magnitude over $\Gamma_t$ is greater than $Y_0C$. The analysis also allows one to consider subspaces of external loadings, e.g., the collection of loadings acting on a particular subset of the boundary.

The load capacity ratio depends only on the geometry of the body and we prove (see [1]) that it is given by

$$\frac{1}{C} = \sup_{w \in LD(\Omega)_D} \frac{\int_{\Gamma_t} |w| \, dA}{\int_{\Omega} |\varepsilon(w)| \, dV} = \| \gamma_D \|.$$ 

Here, $LD(\Omega)_D$ is the space of incompressible integrable vector fields $w$ that satisfy the boundary conditions on $\Gamma_0$ and for which the corresponding stretchings, or linear strains, $\varepsilon(w)$ are assumed to be integrable. This vector space is equipped with the norm

$$\| w \| = \int_{\Omega} |\varepsilon(w)| \, dV$$

making it a Banach space. The norm $|\varepsilon(w)|$ on the space of incompressible, or zero-trace, strain matrices should be chosen as the dual of a norm $|\sigma(x)|$ on the space of deviatoric stress matrices induced by the yield criterion, e.g., the Frobenius norm for von-Mises yield criterion. In addition, $\gamma_D : LD(\Omega)_D \rightarrow L^1(\partial \Omega, \mathbb{R}^3)$ is the trace mapping assigning the boundary value $\gamma_D(w)$ to any $w \in LD(\Omega)_D$. It can be shown (see [2]) that the trace mapping is well defined. Thus, $1/C$ is the operator norm of the trace mapping.

In [3], we present algorithms that enable the application of this theory for computations of the load capacity ratios of structures. We consider trusses, frames and 2-dimensional finite element models of continuous bodies. As an example, consider the plane strain finite element model shown in Figure 1. It is noted that the support is damaged on the left. The strength of the structure to arbitrary loads applied on the right side is considered. In addition to the evaluation of the load capacity ratio of the structure, we compute on the basis of the theory alone a worst case loading distribution to which the structure is most sensitive. Figure 2(a) shows such a worst case loading distribution and Figure 2(b) shows a virtual displacement that maximizes the expression for $1/C$ as above.

The notion of load capacity ratio is a result of the study of optimal stress fields in bodies. Let $t$ be a traction loading on $\Omega$. The stress field in the body should satisfy the equilibrium equations $\sigma_{i,j} = 0$ and boundary conditions $\sigma_{i,j}n_j = t_i$. Without specifying any constitutive relation, there is a subspace $\Sigma_t$ of stress fields that satisfy the equilibrium equations. Engineering stress analysis is traditionally concerned with the
maximum over $\Omega$ of $Y(\sigma(\chi))$, where $Y$ is a norm or a semi-norm on the space of matrices describing the failure criterion for the material that makes up the body. In fact, for perfectly plastic materials, let $\pi(\tau) = \tau - \tau_{ii}I/3$, be the projection of the space of matrices on the space of matrices with zero trace, then, $Y(\tau) = |\pi(\tau)|$, for some norm $|\cdot|$ on the space of matrices. Thus, we consider the optimization problem $s^\text{opt}_t = \inf_{\sigma \in \Sigma} \| Y \circ \sigma \|^\infty$.

It is shown that

$$s^\text{opt}_t = \sup_{w \in L^D(\Omega),D} \frac{\int_{\Omega} |t \cdot w| \, dA}{\int_{\Omega} |\varepsilon(w)|^* \, dV}.$$ 

Furthermore, it turns out that an optimal stress field is attained in a perfectly plastic body if $s^\text{opt}_t = Y_0$. Thus, in the sense described above, perfectly plastic materials are optimal.

**REFERENCES**