Adaptive Filtered Schemes for First Order Evolutive Hamilton-Jacobi Equations: Convergence and Applications

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ABSTRACT

The accurate numerical solution of Hamilton-Jacobi (HJ) equations is a challenging topic of growing importance in many fields of application, e.g. control theory, KAM theory, image processing and material science. This is a delicate issue due to the lack of regularity of viscosity solutions and the construction of high-order methods can be rather complicated. It is well known that simple monotone schemes are at most first order accurate so monotonicity should be abandoned and the proof of high-order convergence becomes very challenging [4] . We propose and analyze a new adaptive filter scheme and prove its convergence to the viscosity solution of the scalar evolutive Hamilton-Jacobi equation

$$\begin{cases} v_t + H(v_x) = 0, & (t,x) \in [0,T] \times \mathbb{R}, \\ v(0,x) = v_0(x), & x \in \mathbb{R}, \end{cases} \tag{1}$$

where Hamiltonian H and the initial data v_0 are Lipschitz continuous functions. A precise result of existence and uniqueness in the framework of weak viscosity solutions can be found in [2]. Our goal is to present a rather simple way to construct high-order schemes for the viscosity solution v of (1) and to prove their convergence at least in the one dimensional case.

In recent years a general approach to the construction of high-order methods using filters has been proposed by Lions and Souganidis in [8] and further developed by Oberman and Salvador [9]. Let us remind that a typical feature of a filtered scheme S^F is that at the node x_j the scheme combines of a high-order scheme S^A and a monotone scheme S^M according to a filter function F. The scheme is written as

$$u_j^{n+1} \equiv S^F(u^n)_j := S^M(u^n)_j + \varepsilon \Delta t F\left(\frac{S^A(u^n)_j - S^M(u^n)_j}{\epsilon \Delta t}\right), \quad j \in \mathbb{Z}, \tag{2}$$

where $\varepsilon = \varepsilon_{\Delta t, \Delta x} > 0$ *is a fixed parameter* going to 0 as $(\Delta t, \Delta x)$ is going to 0 and does not depend on n. Filtered schemes are high-order accurate where the solution is smooth, monotone otherwise, and this feature is crucial to prove a convergence result as in [3]. It is important to note that the choice of ε is delicate since it plays a crucial role in the switching (see [3] for a detailed discussion of this point). Then it seems natural to adapt its choice to the regularity of the solution in the cell via a smoothness indicator improving the filtered scheme (2) by an *adaptive and automatic choice of the parameter* $\varepsilon = \varepsilon^n$ *at every iteration*. Here we introduce a smoothness indicator to select the regions where we have to update the regularity threshold ε^n according to the analysis proposed in [7] although other proposals with similar properties can be applied (see also [1]).

We present a convergence result and some error estimates for the new adaptive filtered scheme in 1D [6] and a 2D application to the segmentation problem in image processing [5].

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