Estudios en la Zona no Saturada del Suelo. Vol IX, O. Silva et al.
Barcelona, 18 a 20 de Noviembre, 2009

PUMPING TEST DETERMINATION OF UNSATURATED AQUIFER PROPERTIES

Phoolendra Kumar Mishra and Shlomo P. Neuman*

Department of Hydrology and Water Resources
University of Arizona
Tucson, Arizona 85721
e-mail: neuman@hwr.arizona.edu

Keywords: Pumping tests, unconfined aquifer, unsaturated zone, properties

ABSTRACT. *We present an analytical solution for flow to a partially penetrating well in a compressible unconfined aquifer that allows inferring its unsaturated hydraulic properties from drawdowns recorded in the saturated zone. Tartakovsky and Neuman (2007) developed such a solution considering an unsaturated zone of infinite thickness. In their solution three-dimensional, axially symmetric unsaturated flow was described by a linearized version of Richards' equation in which both relative hydraulic conductivity and water content vary exponentially with incremental capillary pressure head relative to its air entry value* ψ_a*. Both exponential functions were characterized by a single exponent* κ *having the dimension of inverse length, or equivalently a dimensionless exponent* $\kappa_D = \kappa b$ *where b is initial saturated thickness. We generalize their solution by characterizing relative hydraulic conductivity and water content using different exponential functions and allowing the unsaturated zone to have finite thickness. Our four-parameter representation of unsaturated aquifer properties is more flexible than the three-parameter version of Mathias and Butler (2006), who consider flow in the unsaturated zone to be strictly vertical and the pumping well to be fully penetrating. After validating our solution against numerical simulations of drawdown in a synthetic aquifer having unsaturated properties described by the van Genuchten (1980) and Mualem (1976) models, we investigate the effects of unsaturated zone thickness and constitutive parameters on drawdowns in the unsaturated and saturated zones as functions of position and time. We conclude by using our solution to analyze drawdown data from a pumping test conducted by Moench et al. (2001) in a Glacial Outwash Deposit at Cape Cod, Massachusetts, and compare our results with those of Tartakovsky and Neuman (2007).*

1. INTRODUCTION

We present an analytical solution for flow to a well in an unconfined aquifer that allows inferring its unsaturated hydraulic properties from drawdowns recorded in the saturated zone.

Tartakovsky and Neuman (2007) developed an analytical solution for flow to a partially penetrating well pumping at a constant rate from a compressible unconfined aquifer considering an unsaturated zone of infinite thickness. In their solution three-dimensional, axially symmetric unsaturated flow was described by a linearized version of Richards' equation in which both relative hydraulic conductivity and water content vary exponentially with incremental capillary pressure head relative to its air entry value $\psi_a \geq 0$. Both exponential functions were characterized by a common exponent κ having the dimension of inverse length, or equivalently a dimensionless exponent $\kappa_D = \kappa b$ where *b* is initial saturated thickness.

A solution admitting two separate values of κ, one characterizing relative hydraulic conductivity and the other water content, was developed by Mathias and Butler (2006). Whereas their solution allowed the unsaturated zone to have finite thickness, it considered flow in the unsaturated zone to be strictly vertical and the pumping well to be fully penetrating.

Analyses by Moench (2008) have indicated a need to characterize relative hydraulic conductivity and water content by two separate exponents. The work of Tartakovsky and Neuman (2007) has demonstrated the importance of partial penetration and the existence of horizontal unsaturated flow toward the pumping well. This has led Moench (2008) to conclude that extending the model of Tartakovsky and Neuman (2007) to include two separate exponents, finite unsaturated zone thickness and borehole storage would constitute a welcome addition to the aquifer-test literature. We present an analytical solution that is similar in all respects to that of Tartakovsky and Neuman while characterizing relative hydraulic conductivity and water content by means of separate κ and ψ_a values and taking the thickness of the unsaturated zone to be finite. Our four-parameter representation of these functions is more flexible than the three-parameter version of Mathias and Butler (2006), providing improved fits to standard models such as that of van Genuchten (1980) and Mualem (1976). Our solution further differs from that of Mathias and Butler (2006) in that it allows flow in the unsaturated zone to take place horizontally and the pumping well to be partially penetrating. After validating our solution against numerical simulations of drawdown in a synthetic aquifer having unsaturated properties described by the van Genuchten (1980) and Mualem (1976) models, we investigate the effects of unsaturated zone thickness and constitutive parameters on drawdowns in the unsaturated and saturated zones as functions of position and time. We conclude by using our solution to analyze drawdown data from a pumping test conducted by Moench et al. (2001) in a Glacial Outwash Deposit at Cape Cod, Massachusetts, and compare our results with those of Tartakovsky and Neuman (2007).

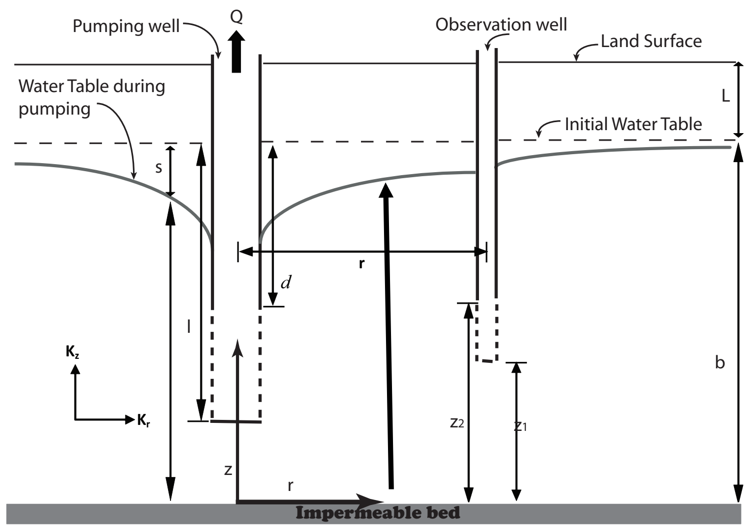

Figure 1. Schematic representation of system geometry.

2. THEORY

2.1. Statement of Problem

In a manner similar to Tartakovsky and Neuman (2007) we consider a compressible unconfined aquifer of infinite lateral extent resting on an impermeable boundary (Figure 1). The aquifer is spatially uniform and anisotropic with a fixed ratio $K_D = K_z / K_r$ between vertical and horizontal saturated hydraulic conductivities, K_z and K_r, respectively. The aquifer is saturated beneath an initially horizontal water table at elevation $z = b$ defined as a $\psi = \psi_a$ isobar where ψ is pressure head and $\psi_a \le 0$ is the pressure head required for air to enter a saturated medium. A saturated capillary fringe at non-positive pressure $\psi_a \le \psi \le 0$ extends from the water table down to the $\psi = 0$ isobar (traditional water table) at elevation $b + \psi_a$ (note that the capillary fringe disappears if one sets $\psi_a = 0$). Prior to the onset of pumping the saturated and overlying unsaturated zones are at uniform initial hydraulic head $h_0 = b + \psi_a$. Starting at time t = 0, water is withdrawn at a constant volumetric rate Q from a well of zero radius that penetrates the saturated zone between depths l and d below the initial water table (at air entry pressure). Under these conditions the drawdown $s(r,z,t)=h_0-h(r,z,t)$ in the saturated zone is governed by the diffusion equation

$$K_r\frac{1}{r}\left(r\frac{\partial s}{\partial r}\right)+K_z\frac{\partial^2 s}{\partial r^2}=S_s\frac{\partial s}{\partial t}\qquad 0\le z\le\xi \tag{1}$$

subject to initial condition

$$\xi(r,0)=b\qquad\text{or}\qquad s(r,z,0)=s_0=0 \tag{2}$$

and boundary conditions

$$s(\infty,z,t)=0 \tag{3}$$

$$\frac{\partial s}{\partial z}=0\qquad z=0 \tag{4}$$

$$\lim_{r\to 0} r\frac{\partial s}{\partial r}=0\qquad 0\le z\le b-l\qquad b-d\le z\le\xi \tag{5}$$

$$\lim_{r\to 0}\int_{b-l}^{\min(b-d,\xi)} r\frac{\partial s}{\partial r}dz=-\frac{Q}{2\pi K_r}\qquad b-l\le z\le b-d \tag{6}$$

where S_s is specific storage and $\xi=b-s(r,\xi,t)$ is head at the water table. If the top of the unsaturated zone is at elevation $z=b+L$ (L being the thickness of the unsaturated zone) then the drawdown $\sigma(r,z,t)=h_0-h(r,z,t)=b+\psi_a-h(r,z,t)$ in this zone is controlled by Richards' equation

$$K_r\frac{1}{r}\frac{\partial}{\partial r}\left(k(\psi)r\frac{\partial\sigma}{\partial r}\right)+K_z\frac{\partial}{\partial z}\left(k(\psi)\frac{\partial\sigma}{\partial z}\right)=C(\psi)\frac{\partial\sigma}{\partial t}\qquad \xi<z<b+L \tag{7}$$

subject to initial condition

$$\sigma_o=\sigma(r,z,0)=0 \tag{8}$$

and boundary conditions

$$\sigma(\infty,z,t)=0 \tag{9}$$

$$\frac{\partial\sigma}{\partial z}=0\qquad z=b+L \tag{10}$$

$$\lim_{r\to 0}\frac{\partial\sigma}{\partial r}=0\qquad \xi<z<b+L \tag{11}$$

where $0\le k(\psi)\le 1$ is relative (ratio of actual to saturated) hydraulic conductivity and $C(\psi)\ge 0$ is specific moisture capacity $C(\psi)=d\theta/d\psi$, θ being volumetric water content. The unsaturated and saturated zone flow regimes are coupled by interface conditions representing continuity of pressure and normal flux across the water table

$$s=\sigma\qquad z=\xi \tag{12}$$

$$\nabla s\cdot\mathbf{n}=\nabla\sigma\cdot\mathbf{n}\qquad z=\xi \tag{13}$$

where $\nabla = (\partial / \partial r, \partial / \partial z)^T$ is the three-dimensional axially-symmetric gradient operator, the superscript T denoting transpose, and **n** is a unit vector normal to the water table.

2.2. Linearization

The above system of equations is highly non-linear due to (a) the nonlinear nature of Richards' equation and (b) the presence of a moving interface (water table) between two different (saturated and unsaturated) flow regimes. To solve it we restrict ourselves to a pumping rate Q that is small compared to $K_r b^2$, expand the dependent variables in power series and disregard terms of order higher than first in Q, as did Kroszynski and Dagan (1975). Substituting the remaining linear terms (in Q) into (1) – (13) and treating horizontal flux into the pumping well as if it was vertically uniform led Tartakovsky and Neuman (2007) to the following first order representation of flow in the saturated zone

$$K_r \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial s}{\partial r}\right) + K_z \frac{\partial^2 s}{\partial z^2} = S_s \frac{\partial s}{\partial t} \qquad 0 \le z < b \tag{14}$$

$$\xi(r,0) = b \qquad \text{or} \qquad s_0 = 0 \tag{15}$$

$$s(\infty, z, t) = 0 \tag{16}$$

$$\frac{\partial s}{\partial z} = 0 \qquad z = 0 \tag{17}$$

$$\lim_{r\to 0} r\frac{\partial s}{\partial r} = 0 \qquad 0 \le z \le b-l \qquad b-d \le z \le b \tag{18}$$

$$\lim_{r\to 0} r\frac{\partial s}{\partial r} = -\frac{Q}{2\pi K_r (l-d)} \qquad b-l \le z \le b-d \tag{19}$$

The corresponding first-order (linearized) unsaturated flow equations are

$$K_r k_0(z)\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \sigma}{\partial r}\right) + K_z \frac{\partial}{\partial z}\left(k_0(z)\frac{\partial \sigma}{\partial z}\right) = C_0(z)\frac{\partial \sigma}{\partial t} \qquad b < z < b+L \tag{20}$$

$$k_0(z) = k(\theta_0) \qquad C_0(z) = C(\theta_0) \tag{21}$$

$$\sigma_0 = 0 \tag{22}$$

$$\sigma(\infty, z, t) = 0 \tag{23}$$

$$\frac{\partial \sigma}{\partial z} = 0 \qquad z = b+L \tag{24}$$

$$\lim_{r\to 0} r\frac{\partial \sigma}{\partial r} = 0 \qquad b < z < b+L \tag{25}$$

The linearized interface conditions at the water table are

$$s - \sigma = 0 \qquad z = b \tag{26}$$

$$\frac{\partial s}{\partial z} - \frac{\partial \sigma}{\partial z} = 0 \qquad z = b \tag{27}$$

Like Tartakovsky and Neuman (2007) we represent the medium water retention characteristics by means of an exponential function

$$S_e = \frac{\theta(\psi) - \theta_r}{S_y} = e^{a_c(\psi - \psi_a)} \qquad a_c \geq 0 \tag{28}$$

where S_e is effective saturation, θ_r is residual water content and $S_y = \theta_s - \theta_r$ is drainable porosity or specific yield. Like them we adopt Gardner's (1958) exponential model for relative hydraulic conductivity

$$k(\psi) = \begin{cases} e^{a_k(\psi - \psi_k)} & \psi \geq \psi_k \\ 1 & \psi_k < \psi \end{cases} \qquad a_k \geq 0 \tag{29}$$

but with parameters a_k and ψ_k that may differ from a_c and ψ_a in (28). The parameter $\psi_k \leq 0$ represents a pressure head above which relative hydraulic conductivity is effectively equal to unity. Our four-parameter representation of these functions is thus more flexible than the two-parameter representations of Tartakovsky and Neuman (2007) (a_c and ψ_a) or Mathias and Butler (2006) (a_c and a_k). It implies that

$$k_0(z) = e^{a_k(b_1 + b - z)} \qquad b_1 = \psi_a - \psi_k \tag{30}$$

$$C_0(z) = S_y a_c e^{a_c(b - z)} \tag{31}$$

in (20) – (21).

2.3. Point Drawdown in Saturated and Unsaturated Zones

In a manner analogous to Neuman (1974) and Tratakovsky and Neuman (2007) we decompose drawdown in the saturated zone into two parts

$$s = s_H + s_U \tag{32}$$

where s_H is Hantush's (1964) solution for a partially penetrating well in a confined aquifer and s_U is a correction due to the presence of an unsaturated zone. Hantush's solution further decomposes into

$$s_H = s_T + \Delta s \tag{33}$$

where s_T is the Theis (1935) solution and Δs is a correction due to partial penetration. This allows obtaining a complete solution in Laplace transformed form. The corresponding time domain solutions s_U and σ are evaluated numerically using the inverse Laplace algorithm of Crump (1976) as modified by de Hoog et al. (1982). The time-domain solution is expressed in terms of dimensionless parameters $K_D = K_z / K_r$, $t_s = \alpha_s t / r^2$, $\alpha_s = K_r / S_s$, $S_D = S_y / S$, $\psi_{kD} = \psi_k / b$, $\psi_{aD} = \psi_a / b$, $a_{kD} = a_k b$, $a_{cD} = a_c b$, $l_D = l / b$, $d_D = d / b$, $r_D = r / b$, and $z_D = z / b$.

3. PREDICTED DRAWDOWN BEHAVIOR

To investigate drawdown behavior based on our new analytical solution we consider, for simplicity, the case where $\psi_a = \psi_k$ and correspondingly $b_1 = 0$.

3.1. Time Drawdown Behavior in Saturated Zone

To investigate the effect of the constitutive exponents a_{kD} and a_{cD} we start by considering an isotropic aquifer $(K_D = 1)$ with a ratio $S / S_y = 1/100$ between artesian storativity and specific yield. To investigate the effects of a_{kD}, the dimensionless exponent for relative hydraulic conductivity, we set the dimensionless exponent for effective saturation to 1.0. We also set the unsaturated zone thickness to infinity, $L \to \infty$. Figure 2 shows how dimensionless drawdown $s_D = 4\pi K_r bs / Q$ varies with dimensionless time $t_s = K_r bt / \left(Sr^2\right)$ on log-log scale at dimensionless elevation $z / b = 0.5$ and dimensionless radial distance $r / b = 0.5$ from the pumping well, which fully penetrates the initial saturated zone. When $a_{kD} = a_{cD} = 1$, our solution reduces to that of Tartakovsky and Neuman (2007). As a_{kD} increases the hydraulic conductivity in the unsaturated zone starts to decrease at a relatively fast rate with a decrease in capillary pressure and hence our solution starts to deviate from that of Tartakovsky and Neuman. At very large a_{kD} this decrease is almost instantaneous and the unsaturated zone behaves as if it was impermeable. Hence the aquifer behaves as if it was confined and our solution reduces to that of Hantush (1964) for a confined aquifer.

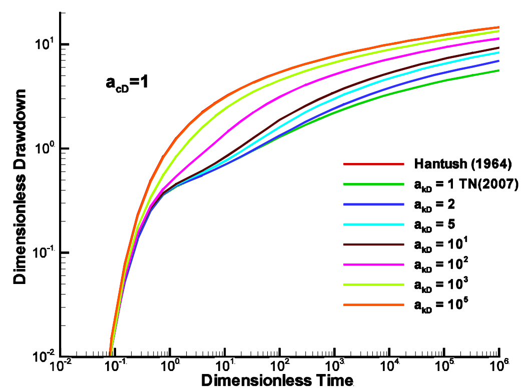

Figure 2. Dimensionless drawdown versus dimensionless time for various values of a_{kD}.

Figure 3 depicts dimensionless time-drawdown behavior on log-log scale when the dimensionless effective saturation exponent a_{cD} varies while a_{kD} is held constant, all other conditions being the same as in Figure 2. When both exponents are large, the unsaturated zone plays virtually no role and our solution reduces to that of Neuman (1974) for a moving free surface.

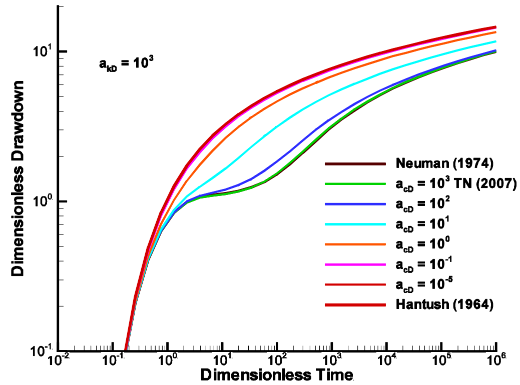

Figure 3. Dimensionless drawdown versus dimensionless time for various values of a_{cD}.

We conclude this section by showing in Figure 4 how time-drawdown is affected by variations in dimensionless unsaturated zone thickness $L_D = L/b$ when $a_{kD} = a_{cD} = 10$ under conditions similar to those in Figures 2 and 3. Reducing L_D is seen to cause dimensionless drawdown at intermediate and late dimensionless times to increase.

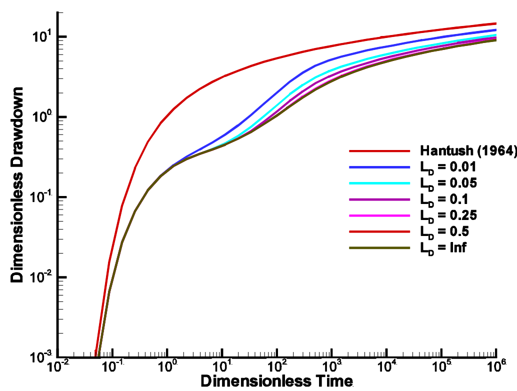

Figure 4. Dimensionless drawdown versus dimensionless time for various values of L_D.

4. ANALYSIS OF SYNTHETIC AQUIFER TEST

We start by considering a 9 m thick anisotropic aquifer $(K_D = 0.4)$ with horizontal hydraulic conductivity, $K_r = 5.0 \times 10^{-3}$ m/s specific storage $S_s = 3.0 \times 10^{-4}$ m^{-1} and specific yield $S_y = 0.322$. Initially, a static water table is situated 2 m below the ground surface. A pumping well discharging at a rate of 60 l/min penetrates the upper 50% of the saturated zone such that $d_d = 0.0$ and $l_d = 0.5$. Water retention and relative hydraulic conductivity are described by the constitutive models of van Genuchten (1980) and Mualem (1976) with parameters $\log \alpha = -1.453$, $\theta_s = 0.375$, $\theta_r = 0.053$ and $\log n = 0.502$ typical of sandy soils (Schaap et al.

2001). Figure 5 shows a least squares fit of our four-parameter exponential models to the latter, yielding parameter estimates $\psi_a = 12$ cm, $\psi_k = 7$ cm, a_k = 8.1 m^{-1}, a_c = 2.9 m^{-1}. Figure 6 compares time drawdowns at dimensionless elevation $z/b = 0.5$ and dimensionless radial distance $r/b = 0.5$ obtained numerically with the STOMP code (Ward et al. 2005) based on van Genuchten – Mualem parameters and our analytical solution with the above best-fit parameters. The agreement is good.

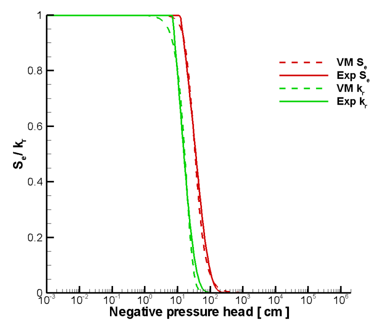

Figure 5. Best fit of our four parameter exponential model to van Genuchten (1980) and Mualem (1976) constitutive models in synthetic case

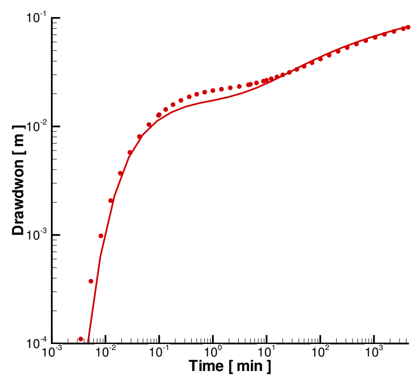

Figure 6. Comparison of numerical and analytical time drawdowns at $z/b = 0.5$ and $r/b = 0.5$ in synthetic case.

5. ANALYSIS OF CAPECOD AQUIFER TEST

We used our solution to analyze drawdown data from a pumping test conducted by Moench et al. (2001) in a Glacial Outwash Deposit at Cape Cod, Massachusetts. We present below preliminary results for 10 observation wells and piezometers lying closest to the pumping well at radial distances not exceeding 26 m. The 10 time-drawdown records were fitted simultaneously to our analytical solution by minimizing the sum of squared differences between them using PEST. A comparison between observed and computed drawdowns corresponding to these 10 records is presented in Figure 7. Table 1 lists the estimated parameters together with those obtained by Moench et al. (2001) and Tartakovsky and Neuman (2007) based on all recorded drawdowns.

Whereas our estimates of hydraulic conductivity and specific storage are similar to those obtained by Moench et al. (2001) and Tartakovsky and Neuman (2007), our specific yield is higher.

The exponential constitutive model parameters $a_c = 0.22$ m^{-1}, $a_k = 1.30$ m^{-1} and $\psi_k - \psi_c = 36.41$ cm in the last raw of Table 1 correspond to van Genuchten (1980) and Mualem (1976) model parameters $\alpha = 0.059$ m^{-1} and $n = 2.14$; the two sets of models are compared in Figure 8.

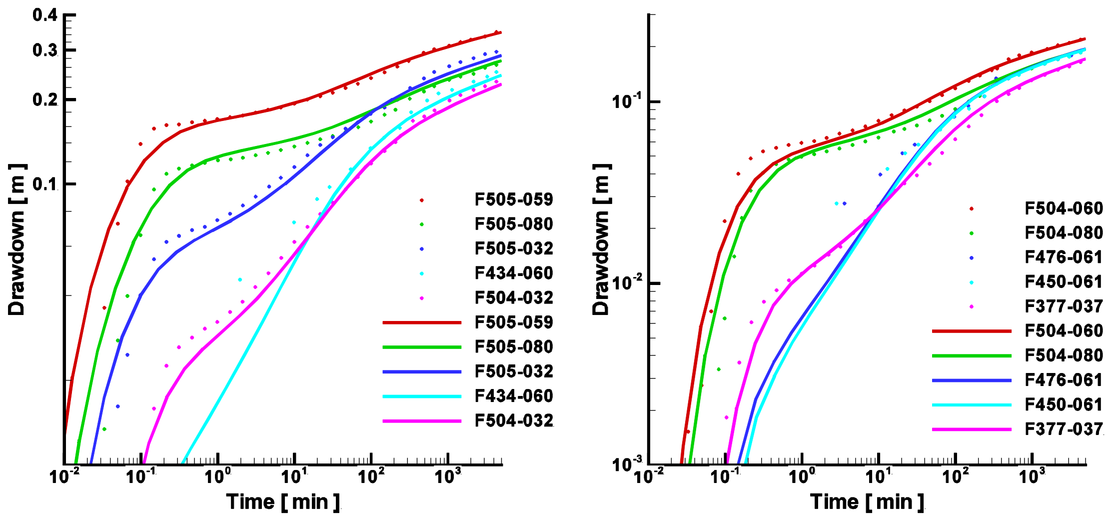

Figure 7. Comparison of simulated (solid lines) and observed drawdowns (dots) at 10 observation wells.

Table 1. Parameters estimates obtained by fitting our solution to 10 drawdown records from observation wells and piezometers closest to the pumping well compared with corresponding estimates by Moench et al. (2001) and Tartakovsky and Neuman (2007).

	K_r (m/s)	K_z (m/s)	S_s (m^{-1})	S_y	a_c (m^{-1})	a_k (m^{-1})	ψ_k - ψ_c (cm)
Moench et al. (2001)	1.17×10^{-3}	7.11×10^{-4}	4.27×10^{-5}	0.26	-	-	-
Tartakovsky and Neuman (2007)	1.02×10^{-4}	8.13×10^{-4}	9.84×10^{-5}	0.18	$\kappa = a_c = a_k = 0.159$		-
This study	1.23×10^{-3}	6.33×10^{-4}	1.04×10^{-4}	0.33	0.22	1.30	36.41

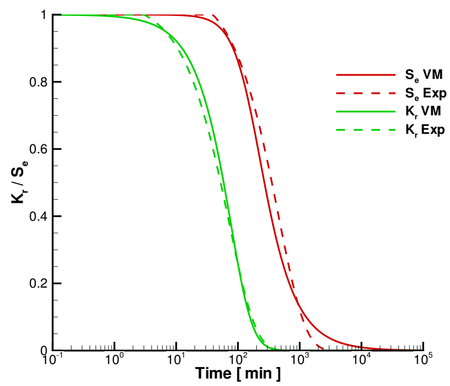

Figure 8. van Genuchten (1980) and Mualem (1976) constitutive models (solid) fitted to exponential constitutive models with parameters in raw 3 of Table 1.

Acknowledgements. This research was supported in part through a contract between the University of Arizona and Vanderbilt University under the Consortium of Risk Evaluation with Stakeholder Participation (CRESP) III, funded by the U.S. Department of Energy.

REFERENCES

Crump, K.S., 1976. Numerical inversion of Laplace transforms using a Fourier series approximation, *J. Assoc. Comput. Mach. 23*, 89–96.

de Hoog, F.R., Knight, J.H., and Stokes, A.N., 1982. An improved method for numerical inversion of Laplace transforms, *SIAM J. Sci. Comput. 3*, 357–366, doi:10.1137/0903022.

Gardner, W.R., 1958. Some steady state solutions of unsaturated moisture flow equations with application to evaporation from a water table, *Soil Sci. 85*, 228–232.

Hantush, M.S., 1964. Hydraulics of wells, *Adv. Hydrosci. 1*, 281–442.

Kroszynski, U.I., and Dagan, G., 1975. Well pumping in unconfined aquifers: The influence of the unsaturated zone, *Water Resour. Res. 11*, 479–490, doi:10.1029/WR011i003p00479.

Mathias, S.A., and Butler, A.P., 2006. Linearized Richards' equation approach to pumping test analysis in compressible aquifers, *Water Resour. Res. 42*, W06408, doi:10.1029/2005WR004680.

Moench, A.F., Garabedian, S.P., and LeBlanc, D.R., 2001. Estimation of hydraulic parameters from an unconfined aquifer test conducted in a glacial outwash deposit, Cape Cod, Massachusetts, *U.S. Geol. Surv. Prof. Pap. 1629*, 69 pp.

Moench, A.F., 2008. Analytical and numerical analyses of an unconfined aquifer test considering unsaturated zone characteristics, *Water Resour. Res. 44*, W06409, doi:10.1029/2006WR005736.

Mualem, Y., 1976. A new model for predicting the hydraulic conductivity of unsaturated porous media, *Water Resour. Res. 12*(3), 513 – 522.

Neuman, S.P., 1974. Effects of partial penetration on flow in unconfined aquifers considering delayed aquifer response, *Water Resour. Res. 10*, 303–312, doi:10.1029/WR010i002p00303.

Schaap, M.G., Leij, F.J., and van Genuchten, M.Th., 2001. Rosetta: a computer program for estimating soil hydraulic parameters with hierarchical pedotransfer functions, *Journal of Hydrology 251*, 163-176.

Tartakovsky, G.D., and Neuman, S.P., 2007. Three-dimensional saturated-unsaturated flow with axial symmetry to a partially penetrating well in a compressible unconfined aquifer, *Water Resour. Res. 43*, W01410, doi:10.1029/2006WR005153.

Theis, C.V., 1935. The relationship between the lowering of the piezometric surface and rate and duration of discharge of a well using groundwater storage, *Eos Trans. AGU 16*, 519–524.

van Genuchten, M.T., 1980. A closed-form equation for predicting the hydraulic conductivity of unsaturated soils, *Soil Sci. Soc. Am. J. 44*, 892–898.

Ward, A.L., White, M.D., Freeman, E.J., and Zhang, Z.F., 2005. STOMP Subsurface Transport Over Multiple Phase, Addendum: Sparse Vegetation Evapotranspiration Model for the Water-Air-Energy Operational Mode, PNNL 15465, Pacific Northwest National Laboratory, Richland, Washington.