Helmholtz equation in highly heterogeneous media

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The acoustic Helmholtz equation is used by geoscientists to model the propagation of seismic waves through the Earth. Currently, each material composing the subsurface is represented by a wave velocity and the equation for the pressure reads

$$-\frac{\omega^2}{c^2}u - \Delta u = f,$$

where c is the (heterogeneous, varying in space) velocity parameter. In the process of solving Full Waveform Inversion problems, many solutions to the Helmholtz equation need to be numericaly computed for different frequencies ω and different topologies represented by the parameter c. This motivates the implementation of efficient approximation methods for the Helmholtz equation in heterogeneous media witch are adapted to both high frequencies ω and complex velocity parameters c.

In the homogeneous case (i.e. c is constant), it is well-known that when the frequency increases, the Helmholtz operator looses ellipticity which enforces drastic conditions on the mesh to ensure discrete stability. This is the so-called pollution effect. In the case of star-shaped domains, a wave-number explicit stability theory of the continuous problem is available [1]. It has been shown that $|u|_{l,\Omega} \leq C\omega^{l-1}|f|_{0,\Omega}$ for $l = 0, 1, 2$. Regarding discretization, frequency splitting arguments have been used in [2], to prove that high order methods are adapted to handle pollution, and stability is ensured under the condition that $\omega^{p+1}h^p$ are small enough (where h is the discretisation step and p is the order of discretisation). This methodology leads to coarse meshes, with several degrees of freedom per cell.

From a numerical point of view, high order methods are also known to be efficient in heterogeneous media. However, they fail to handle small heterogeneities. Indeed, the use of coarse meshes implies that the velocity parameter may change within a cell and two issues have to be considered: the solution is less regular within a cell and analytic computations of the Finite Element linear system coefficients is no longer possible.

In this work, we show that the stability results for star-shaped domains may be extended to the heterogeneous case when the velocity parameter c is monotonous. Then, we show that high order methods with coarse meshes may capture very small heterogeneities if adapted quadrature formula are used to compute the coefficients of the linear system. If H denotes the mesh step, h stands for the quadrature step and $1 \leq p \leq 3$ is the discretisation order, we prove that the discrete system is stable if $\omega^2 Hh$ and $\omega^{p+1}H^p$ are small enough. Numerical experiments illustrate the efficiency of our approach.

References

[1] U. Hetmaniuk. Stability estimates for a class of helmholtz problems. *COMMUN. MATH. SCI.*, 2007.

[2] J.M. Melenk and S. Sauter. Wavenumber explicit convergence analysis for galerkin discretisations of the helmholtz equation. *SIAM J. NUMER. ANAL.*, 2011.