Relaxing mixed integer optimal control problems using a time transformation

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Abstract

Nonlinear control systems with instantly changing dynamic behavior can be described by differential equations $\dot{x} = F(x,u,v)$ that depend on an integer valued control function $v \in \mathscr{L}^\infty(I,\mathscr{V})$, mapping the time interval $I = [t_0, t_f]$ to the integer values $\mathscr{V} = \{1,\ldots,n_\mathscr{V}\}$. Such systems occur for example in the optimal control of a driving car with different gears [1], or a subway ride with different operation modes [2], leading to a mixed integer optimal control problem (MIOCP). A discretize-then-optimize approach leads to a mixed integer optimization problem (MIOP) that is not differentiable with respect to the integer variables, such that gradient based optimization methods can not be applied. Differentiability with respect to all optimization variables can be achieved by reformulating the MIOCP, e.g. by using a relaxed binary control function [2], or by introducing a fixed integer control function $\bar{v}_{N,n} \in \mathscr{L}^\infty(I,\mathscr{V})$ and a time transformation $t_w \in \mathscr{W}^{1,\infty}(I,I)$ that allows to partially deactivate the fixed integer control function [1]. The latter approach is presented here, while the main focus lies on new theoretical and numerical results that take different functions $\bar{v}_{N,n}$ into account. The time interval I is partitioned into N major intervals I_j and each $I_j = [t_{j-1}, t_j[$ into n minor intervals $I_j^i = [\tau_j^{i-1}, \tau_j^i[$. Then, the function $\bar{v}_{N,n}$ is defined to be constant on each minor interval I_j^i. Figure 1 depicts an example of such a fixed integer control function. The depicted function is called *consistent* to every integer control function v, because a

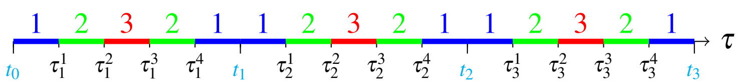

Figure 1: Example of a fixed integer control function $\bar{v}_{N,n}$ with $\mathscr{V} = \{1,2,3\}$ and $N = 3$ major and $n = 5$ minor intervals

switch of v at any time in a major interval I_j from a value $l_1 \in \mathscr{V}$ to a value $l_2 \in \mathscr{V}$ can be achieved with $\bar{v}_{N,n}$ by scaling the minor intervals I_j^i, in particular some minor intervals can be deactivated by scaling to zero. The scaling is accomplished by a time transformation $t_w \in \mathscr{W}^{1,\infty}(I,I)$ resulting from a time control $w \in \mathscr{L}^\infty(I,\mathbb{R})$ with $\Delta I_j = \int_{I_j} w(s)ds$ and $w(\tau) \geq 0$ for a.e. $\tau \in I$. The time transformation with derivative $t'_w(\tau) = \frac{dt_w}{d\tau}(\tau) = w(\tau)$ for a.e. $\tau \in I$ is defined by

$$t_w(\tau) := t_0 + \int_{t_0}^{\tau} w(s)ds, \tag{1}$$

and assures that the mapping of a major interval is surjective $t_w(I_j) = I_j$ even if several minor intervals I_j^i are deactivated, i.e. $t_w(I_j^i)$ has zero length. Then, the time transformed MIOCP is defined as follows:

Definition 1. *(TMIOCP)*

$$
\begin{aligned}
\min_{x,u,w} \quad & J^*(x,u,w) = \int_I w(\tau) B(x(\tau), u(\tau), \bar{v}_{N,n}(\tau))\, d\tau && && (2)\\
s.\ t. \quad & \dot{x}(\tau) = w(\tau) F(x(\tau), u(\tau), \bar{v}_{N,n}(\tau)) && \textit{for a.e. } \tau \in I && (3)\\
& g_0(x(\tau), u(\tau)) \leq 0 && \textit{for a.e. } \tau \in I && (4)\\
& w(\tau) g(x(\tau), u(\tau), \bar{v}_{N,n}(\tau)) \leq 0 && \textit{for a.e. } \tau \in I && (5)\\
& r(x(t_0), x(t_N)) = 0 && && (6)\\
& w(\tau) \geq 0 && \textit{for a.e. } \tau \in I && (7)\\
& \Delta I_j = \int_{I_j} w(s) ds && && (8)
\end{aligned}
$$

Here, J is the objective functional, g_0 and g are inequality constraints and r represents the boundary conditions. An example demonstrates that solving a TMIOCP using a control consistent (CC) fixed integer control function $\bar{v}_{N,n}$ can lead to a lower number of discretization variables as a TMIOCP that utilizes a fixed integer function that is not control consistent (NCC). The number of necessary discretization variables depends on the total number of used minor intervals and it is shown that the total number can be unbounded in the NCC case and is bounded in the CC case, even though the number of minor intervals for each major interval is lower in the NCC case. As a numerical example, a hybrid mass oscillator is considered (Figure 2 (a)), that extends the example given in [3] to three springs. The springs are mounted in parallel and relaxed in different positions. A mass is fixed on the first spring, and the second and third springs are activated and deactivated depending on the position of the mass. A minimization with unspecified end position results in the CC case in the expected oscillating trajectory with almost no control effort. In contrast, the locally optimal trajectory in the NCC case avoids specific switches in the interior of major intervals by oscillating with a low amplitude and needs a high control effort. The state trajectories are plotted in Figure 2 (b) and (c). An extension to bipedal walking models is planed in future works.

(a) hybrid mass oscillator (b) $J^*(u) = 6.0520911696 \cdot 10^{-11}$ (c) $J^*(u) = 3.7453160945 \cdot 10^{8}$

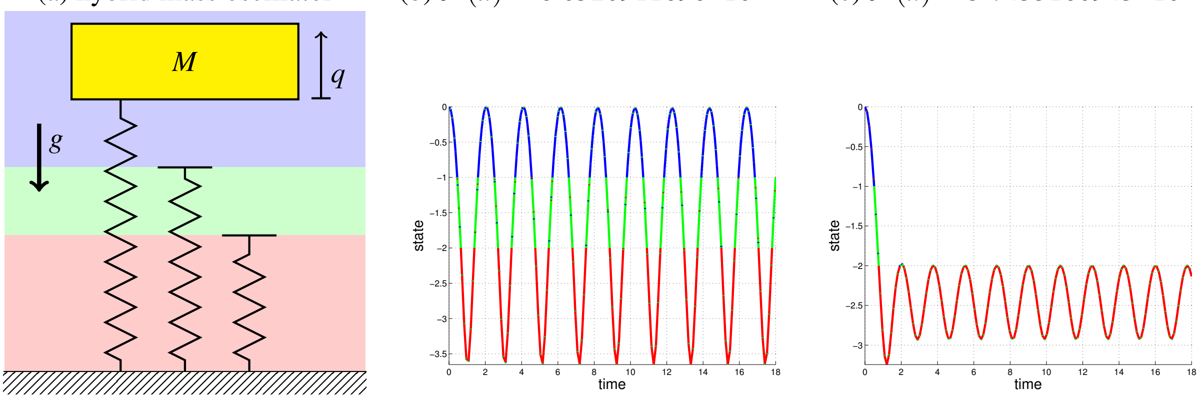

Figure 2: (a) Sketch of the hybrid mass oscillator. (b) Locally optimal discretized state trajectory of the hybrid mass oscillator in the CC case and (c) in the NCC case.

References

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