Contact problems using quaternions in Lagrange Equations

René Souchet

A.F.M.
13 Rue Johann Strauss, Buxerolles,
86180, France
rp.souchet@orange.fr

Abstract
In recent years, authors Udwadia et al. [1] have proposed to obtain dynamical equations using Lagrange method with generalised parameters as quaternions *q*. In 2014, a different point of view was applied by the actual author to treat (friction) problems whatever the nature of the parameters e.g. quaternions. Since rigidity is not included, the msain aim is the necessary use of stress tensor in the Virtual Work Principle (VWP), then its elimination for rigid bodies. Here we propose to show the applicability of our method to an example involving friction expressed by inequality relations. In this paper it is highlighted the fundamental difference between constraints issued from rigidity or friction.

Background. If body forces are not present for simplicity, the VWP is written for a body *B*

$$-\int_B \rho a.v\, dx + \int_\Gamma \varphi.v\, da - \int_B \sigma{:}grad v\, dx = 0$$

where ρ is the density, a the acceleration, φ the surface forces, σ the stress tensor, and v the virtual piecewise displacements. In the present application of some rotational motion, $x=R(q(t))X$, x being the actual position of the particle X, the virtual displacements are $v=(R'_i\ R^{-1}x)w_i$ where the w_i's are arbitrary and R is a 3x3 matrix function of quaternions. R'_I is the partial derivative of $R(q_1,\ldots,q_n)$. R is not necessarily a rotation, i.e. the constraint $q^Tq=1$ is not fulfilled as an a priori condition.

If we take account of the actual virtual displacements in the above formula,, then the first term is the virtual work (denoted L_iw_i) of acceleration, where L_i is obtained by Lagrange usual formula as a function of kinetic energy. Then we have

$$grad\ v= (R'_i\ R^{-1})w_i=S_iw_i+A_iw_i,\quad \sigma{:}grad v=(\sigma{:}S_i)w_i$$

where S_i and A_i are resp. the symmetrical and anti-symmetrical parts of the matrix $R'_i\ R^{-1}$. The last equality does not contain the matrix A_i since σ is symmetric and A_i is anti-symmetric. Now in order to eliminate the stress tensor, we require the relations $S_iw_i=0$.(sum on i), a priori realised if R is a rotation. In addition, it is seen that surface forces f occur by global quantities only (i.e. $R(f)$ and $M(f)$). So the following compatibility conditions result: **whatever the w_i's such that $S_iw_i=0$, we have**

$$[-L_i+M(f)\ a_i]w_i=0 \quad \textbf{(sum on } i\textbf{)}$$

(a_i: dual vector of matrix A_i) under the only above hypotheses.

Finally we write the rigidity constraint (the material constitutive law) $q^Tq=1$ since quaternions are used. It is noteworthy that no undue hypothesis of the virtual work of internal forces was made in our paper. Now we apply these relations to a contact problem.

Example: contact with friction. We consider an homogeneous rigid wheel (centre O, radius r and mass m) rolling in a vertical plane $O_0x_0y_0$ on an inclined line (or surface) O_0X_0 under the gravitational acceleration g downwards, the gravitational force being ($f=-mgy_0$) applied on the centre O of the wheel. We use the referential $Ref=O_0X_0Y_0Z_0$ with the angle between O_0x_0 and O_0X_0 noted a . Two-dimensional Euler parameters (p,q) are introduced to specify the rotation of the wheel, so writing for the matrix R

$$R_{11}=R_{22}=1-2q^2\ ,\ R_{12}=-R_{21}=-2pq\ ,\ R^{-1}=R^T/\Delta \quad ,\quad \Delta=1+4q^2(p^2+q^2-1$$

Now we introduce the virtual coefficients (w_x,w_y,w_p,w_q) associate to the parameters (x,y,p,q) and the condition $w_iS_i=0$, i.e. $pw_p+qw_q=0$. Under the above condition, the VWP is writing

$$-\int_B \rho a.v\,dx - mgy_0 .v(O)+Tv_1(A)+Nv_2(A)=0$$

where *(T,N,0)* are the components of the two-dimensional contact force on the wheel applied at the contact point *A*. Now we must use the contact law of friction, by example in the hypothesis of a bilateral contact *(y=r)* at the point *A=(x,y-r,0)* of the wheel, implying the geometric constraint *y=r* , together with the Coulomb law of friction equivalent to the inequality of Duvaut and Lions

$$T\,[v_1(A)-u_1(A)]+k\,|N|\,[|v_1(A)|-|u_1(A)|]\geq 0$$

First the parameters are specified such that $w_x=w_p=w_q=0$, satisfying $w_iS_i=0$. It results $v(x)=(0,w_y,0)$ so that by taking account of the bilateral contact $y=r$

$$mgcosa-N=0 \text{ and } \dot{K} + mgsina\ \dot{x} + k\,|N|\,|u_1(A)| = 0$$

$$\int_B \rho a.v\,dx + mgy_0\, v(O)-Nv_2(A)+ k\,|N|\,|v_1(A)| \geq 0 \quad \text{where } N=mg\,cosa$$

that is available whatever the parameters (w_x,w_p,w_q). After some straightforward calculus, the acceleration term is obtained under the form

$$\int_B \rho a.v\,dx = m\ddot{x}w_x + a_{11}\dot{p} + a_{22}\dot{q} + 2a_{12}\dot{p}\dot{q} + 2b\dot{q}^2$$

$$a_{11}=2mr^2(q^2w_p+pqw_q),\ a_{22}=2mr^2(pq+p^2+4q^2)w_q,\ a_{12}=2mr^2(qw_p+pw_q),\ b=4mr^2qw_q$$

Taking account of this expression, the differential variational inequality follows

$$(m\ddot{x}+mg\sin a)w_x + kmg\cos a\,|w_x+r(\alpha_p w_p+\alpha_q w_q)| + 2mr^2(Aw_p+Bw_q)\geq 0$$

(where α_p,α_q and *A,B* are given functions) under the compatibility condition $pw_p+qw_q=0$. That is the basic relation to solve the problem completed naturally by initial conditions on velocities (and positions). The numerical treatment of this inequality is not the aim of this present mechanical work.

Conclusion. The present work has presented a natural link existing between Analytical Dynamics and Continuum Mechanics. The key of our scheme was the use of the Virtual Work Principal. Then the elimination of Cauchy stresses introduces compatibility relations between virtual coefficients.

References

[1] Udwadia, F.E., Schutte, A.D., “An Alternative Derivation of the Quaternion Equations of Motion for Rigid-Body Rotational Dynamics”, Transactions of the ASME, Vol. 77 pp. 044505-1, 044505-4, 2010.

[2] Souchet, R., Leçons sur les Grandes Déformations, Cépadues Editions, Toulouse, France, 2004.

[3] Souchet, R.,.Continuum Mechanics and Lagrange equations with generalised coordinates, Int. J. Engng Sc., 76C pp. 27-33, 2014.

[4] Duvaut G., Lions J.L., Les inéquations en Mécanique et en Physique, Dunod, Paris, (chapitre 3, Problèmes classiques et problèmes de frottement en Elasticité et Visco-Elasticité), 1972.