Geometrically based pseudo-inverse and reciprocal screws of the Jacobian of 4H mechanisms

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Abstract

Overconstrained single-loop mechanisms with $n \leq 6$ helical joints (nH mechanisms), see Fig. 1, are mobile with $f = 1$ degree of freedom if the screw axes of the helical joints $\hat{\boldsymbol{a}}_i$ fulfill the first-order loop-closure condition

$$\boldsymbol{A}(s)\,\boldsymbol{\lambda}(s) = \underline{\hat{\boldsymbol{a}}}_n \quad \text{with} \quad \boldsymbol{A} = \left[\underline{\hat{\boldsymbol{a}}}_1 \ldots \underline{\hat{\boldsymbol{a}}}_{n-1}\right], \quad \underline{\hat{\boldsymbol{a}}}_j = \begin{bmatrix} \boldsymbol{a}_j \\ \widetilde{\boldsymbol{r}}_j \boldsymbol{a}_j + h_j \boldsymbol{a}_j \end{bmatrix}, \quad \boldsymbol{\lambda} = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_{n-1} \end{bmatrix}, \quad \lambda_i = -\frac{\mathrm{d}q_i}{\mathrm{d}q_n}, \tag{1}$$

here parametrised, without loss of generality, by the joint coordinate $s = q_n$ of the nth helical joint, in an open neighborhood of an actual position $s_0 = q_{n0}$ of the independent joint coordinate (with the notation $\widetilde{\boldsymbol{a}}\boldsymbol{b} = \boldsymbol{a} \times \boldsymbol{b}$).

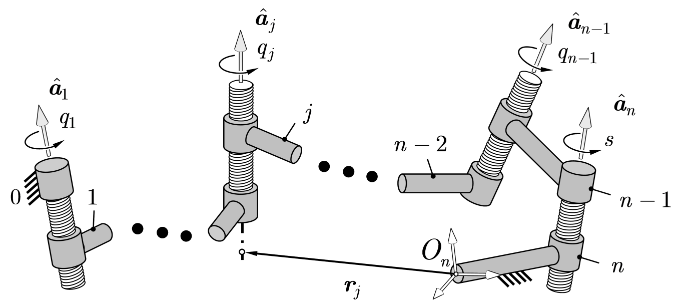

Figure 1: Mechanism with n helical joints (nH mechanism)

Since the closure condition of an nH mechanism is an analytical function, the Taylor-series expansion of (1) yields an infinite number of equivalent algebraical conditions for the mobility of the mechanism, called the closure conditions of order $m = 1, \ldots, \infty$ in the actual position of the mechanism, $s_0 = q_{n0}$. These closure conditions contain the derivatives of the joint screws, $\hat{\boldsymbol{a}}_i, \hat{\boldsymbol{a}}_i', \ldots$, and the ratios of the joint coordinates with respect to the independent joint coordinate, $\boldsymbol{\lambda}_i, \boldsymbol{\lambda}_i', \ldots$. The elimination of the joint coordinates with the help of a pseudo-inverse $\boldsymbol{A}_0^+$ of the Jacobian $\boldsymbol{A}_0$ and the reciprocal screws $\underline{\hat{\boldsymbol{k}}}_{j0}, j = 1, \ldots, (6 - \mathrm{rank}(\boldsymbol{A}_0))$ of the column screws contained in $\boldsymbol{A}_0$, thus (unit matrix $\boldsymbol{I}$)

$$\underline{\hat{\boldsymbol{k}}}_{j0}^{\mathrm{T}} \Delta \boldsymbol{A}_0 \overset{!}{=} 0 \quad \text{with} \quad \Delta = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ \boldsymbol{I} & \boldsymbol{0} \end{bmatrix}, \quad j = 1, \ldots, (7-n), \tag{2}$$

yields a system of necessary mobility conditions, as shown in [2],

$$\left.\begin{aligned}
m = 1: &\quad 0 = \underline{\hat{\boldsymbol{k}}}_{j0}^{\mathrm{T}} \Delta \underline{\hat{\boldsymbol{a}}}_{n0} && \equiv g_1\left(\underline{\hat{\boldsymbol{a}}}_{n0}, \boldsymbol{A}_0\right) \\
m = 2: &\quad 0 = \underline{\hat{\boldsymbol{k}}}_{j0}^{\mathrm{T}} \Delta \boldsymbol{A}_0' \boldsymbol{A}_0^+ \underline{\hat{\boldsymbol{a}}}_{n0} && \equiv g_2\left(\underline{\hat{\boldsymbol{a}}}_{n0}, \boldsymbol{A}_0\right) \\
m = 3: &\quad 0 = \underline{\hat{\boldsymbol{k}}}_{j0}^{\mathrm{T}} \Delta \left(-2\boldsymbol{A}_0' \boldsymbol{A}_0^+ \boldsymbol{A}_0' + \boldsymbol{A}_0''\right) \boldsymbol{A}_0^+ \underline{\hat{\boldsymbol{a}}}_{n0} && \equiv g_3\left(\underline{\hat{\boldsymbol{a}}}_{n0}, \boldsymbol{A}_0\right) \\
&\quad \vdots \\
m: &\quad 0 = \ldots && \equiv g_m\left(\underline{\hat{\boldsymbol{a}}}_{n0}, \boldsymbol{A}_0\right)
\end{aligned}\right\} \quad \boldsymbol{g}_{(m)}\left(\underline{\hat{\boldsymbol{a}}}_{10}, \ldots, \underline{\hat{\boldsymbol{a}}}_{n0}\right) = \boldsymbol{0}. \tag{3}$$

The derivatives of the matrix $\boldsymbol{A}_0 = \left[\ \hat{\underline{\boldsymbol{a}}}_{10} \quad \cdots \quad \hat{\underline{\boldsymbol{a}}}_{n-1,0}\ \right]$ with respect to s are expressed by the derivatives of the screw axes $\hat{\underline{\boldsymbol{a}}}'_{k0} = \frac{\mathrm{d}\hat{\underline{\boldsymbol{a}}}_k}{\mathrm{d}s}|_{q_{n0}}$ using the dual vector product,

$$\hat{\underline{\boldsymbol{a}}}'_{k0} = -\sum_{i=1}^{k-1} \widetilde{\hat{\underline{\boldsymbol{a}}}}_{i0}\,\hat{\underline{\boldsymbol{a}}}_{k0}\,\lambda_{i0} \quad \text{with} \quad \widetilde{\hat{\underline{\boldsymbol{a}}}} \equiv \begin{bmatrix} \widetilde{\boldsymbol{a}} & \boldsymbol{0} \\ \widetilde{\boldsymbol{a}}_\varepsilon & \widetilde{\boldsymbol{a}} \end{bmatrix}. \tag{4}$$

Due to the successively elimination of the joint coordinates, the mobility conditions (3) are nonlinear equations only in the coordinates $\hat{\underline{\boldsymbol{a}}}_{i0}, i = 1,\ldots,n$ of the joint screws in the actual position of the mechanism. The solution of (3) up to an unknown sufficient finite order $m_{\max}$, depending on the number and type of joints, yields these screw coordinates of the joints which guarantee the finite mobility of the mechanism, see also [1]. For several overconstrained mechanisms an estimation of $m_{\max}$ by the numerical solution of (3) was given in [3]. One difficulty is to find a analytical solution of (3) because the pseudo-inverse $\boldsymbol{A}_0^+$ as well as the reciprocal screws $\hat{\underline{\boldsymbol{k}}}_{j0}, j = 1,\ldots,(6-\mathrm{rank}(\boldsymbol{A}_0))$ are not given in an algebraical form. But for 4H mechanisms these expressions can be derived by geometrical considerations. The special pseudo-inverse of the Jacobian $\boldsymbol{A}_0 = \left[\ \hat{\underline{\boldsymbol{a}}}_{10} \quad \hat{\underline{\boldsymbol{a}}}_{20} \quad \hat{\underline{\boldsymbol{a}}}_{30}\ \right]$ given by

$$\boldsymbol{A}^+_{0,\mathrm{geom}} = \beta \left[\ \widetilde{\hat{\underline{\boldsymbol{a}}}}_{20}\,\hat{\underline{\boldsymbol{a}}}_{30} \quad \widetilde{\hat{\underline{\boldsymbol{a}}}}_{30}\,\hat{\underline{\boldsymbol{a}}}_{10} \quad \widetilde{\hat{\underline{\boldsymbol{a}}}}_{10}\,\hat{\underline{\boldsymbol{a}}}_{20}\ \right]^{\mathrm{T}} \Delta, \quad \beta = (\widetilde{\hat{\underline{\boldsymbol{a}}}}_{10}\,\hat{\underline{\boldsymbol{a}}}_{20})^{\mathrm{T}} \Delta \hat{\underline{\boldsymbol{a}}}_{30} \tag{5}$$

contains the three screws $\widetilde{\hat{\underline{\boldsymbol{a}}}}_{20}\,\hat{\underline{\boldsymbol{a}}}_{30}$, $\widetilde{\hat{\underline{\boldsymbol{a}}}}_{30}\,\hat{\underline{\boldsymbol{a}}}_{10}$, $\widetilde{\hat{\underline{\boldsymbol{a}}}}_{10}\,\hat{\underline{\boldsymbol{a}}}_{20}$. The axes of these screws are the common perpendicular lines of two adjacent joint axes. By this, these screws are each reciprocal to two screws of the joint axes $\hat{\underline{\boldsymbol{a}}}_{10}$, $\hat{\underline{\boldsymbol{a}}}_{20}$, $\hat{\underline{\boldsymbol{a}}}_{30}$ of the 4H mechanism. In the same way the three screws

$$\hat{\underline{\boldsymbol{k}}}_{10,\mathrm{geom}} = \widetilde{\hat{\underline{\boldsymbol{a}}}}_{20}\,\hat{\underline{\boldsymbol{a}}}_{30} + \begin{bmatrix} \boldsymbol{0} \\ h^*\widetilde{\boldsymbol{a}}_{20}\boldsymbol{a}_{30} \end{bmatrix},\ \hat{\underline{\boldsymbol{k}}}_{20,\mathrm{geom}} = \widetilde{\hat{\underline{\boldsymbol{a}}}}_{30}\,\hat{\underline{\boldsymbol{a}}}_{10} + \begin{bmatrix} \boldsymbol{0} \\ h^*\widetilde{\boldsymbol{a}}_{30}\boldsymbol{a}_{10} \end{bmatrix},\ \hat{\underline{\boldsymbol{k}}}_{30,\mathrm{geom}} = \widetilde{\hat{\underline{\boldsymbol{a}}}}_{10}\,\hat{\underline{\boldsymbol{a}}}_{20} + \begin{bmatrix} \boldsymbol{0} \\ h^*\widetilde{\boldsymbol{a}}_{10}\boldsymbol{a}_{20} \end{bmatrix}, \tag{6}$$

with

$$h^* = -\frac{1}{\beta\,(\widetilde{\boldsymbol{a}}_{10}\,\boldsymbol{a}_{20})^{\mathrm{T}}\,\boldsymbol{a}_{30}}, \tag{7}$$

are reciprocal to the screws $\hat{\underline{\boldsymbol{a}}}_{10}$, $\hat{\underline{\boldsymbol{a}}}_{20}$, $\hat{\underline{\boldsymbol{a}}}_{30}$. There exist special positions of 4H mechanisms where the geometrically based expressions $\boldsymbol{A}^+_{0,\mathrm{geom}}$ and $\hat{\underline{\boldsymbol{k}}}_{j0,\mathrm{geom}}, j = 1,\ldots,3$ are not valid. These positions have to be excluded as start positions for the local approximation.
In a first step it can be shown for the BENNETT 4R mechanism, that the mobility condition as well as the closure condition of order $m = 3$,

$$\boldsymbol{0} = \left[\ \hat{\underline{\boldsymbol{k}}}_{10,\mathrm{geom}} \quad \hat{\underline{\boldsymbol{k}}}_{20,\mathrm{geom}} \quad \hat{\underline{\boldsymbol{k}}}_{30,\mathrm{geom}}\ \right]^{\mathrm{T}} \Delta\left(-2\boldsymbol{A}'_0\boldsymbol{A}^+_{0,\mathrm{geom}}\boldsymbol{A}'_0 + \boldsymbol{A}''_0\right)\boldsymbol{A}^+_{0,\mathrm{geom}}\,\hat{\underline{\boldsymbol{a}}}_{40}, \tag{8}$$

are fulfilled, if the mobility conditions of order $m = 1$ and $m = 2$,

$$\begin{aligned} \boldsymbol{0} &= \left[\ \hat{\underline{\boldsymbol{k}}}_{10,\mathrm{geom}} \quad \hat{\underline{\boldsymbol{k}}}_{20,\mathrm{geom}} \quad \hat{\underline{\boldsymbol{k}}}_{30,\mathrm{geom}}\ \right]^{\mathrm{T}} \Delta\hat{\underline{\boldsymbol{a}}}_{40} \\ \boldsymbol{0} &= \left[\ \hat{\underline{\boldsymbol{k}}}_{10,\mathrm{geom}} \quad \hat{\underline{\boldsymbol{k}}}_{20,\mathrm{geom}} \quad \hat{\underline{\boldsymbol{k}}}_{30,\mathrm{geom}}\ \right]^{\mathrm{T}} \Delta\boldsymbol{A}'_0\boldsymbol{A}^+_{0,\mathrm{geom}}\,\hat{\underline{\boldsymbol{a}}}_{40} \end{aligned} \tag{9}$$

are fulfilled. This result can be obtained by introducing recursively the mobility conditions together with the properties of screw products, without explicitly solving the closure conditions.

References

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