Aircraft Attitude Dynamics in Terms of Quaternions Using a Non-Redundant ODE Lie-Group Formulation

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Abstract
Dynamic simulation procedures of flight vehicle (fixed-wing, rotorcraft, UAV, satellite) 3D manoeuvres need robust and efficient integration methods in order to allow for reliable, and possibly real-time, simulation missions. Since flight vehicle 3D manoeuvres necessary include complete 3D rotation domain, such procedures also require an efficiency way of dealing with large 3D rotation. Usually, in this context, the simulation procedures built around the standard numerical ordinary-differential-equations (ODE) based on three-parameters rotation variables (such as Euler angels) have their limitations, as they impose discontinuities or even singularities in the flight vehicle attitude integral curves. Most commonly, the quaternion representation is widely used in flight simulation to overcome the mentioned deficiency [1]. However, if quaternions are used for a parameterisation of the rotation manifold, the standard model leads to integration of differential-algebraic equations (DAE) that requires additional stabilisation of the algebraic constraint due to quaternions normalisation equation. Recently, a method of integration of rotational quaternions based on non-vectorial geometric Lie-group integration that leads to a minimal-form ODE integration (avoiding thus DAE integration) has been introduced in [2]. By adopting such an approach, the proposed method is based on numerical integration of the kinematic relations in terms of the instantaneous rotation vector that form an ODE on Lie-algebra *so*(3) of the rotation group *SO*(3), after which the integration incremental update on the configuration quaternion group *Sp*(1) is determined by the exponential map. Consequently, only a system of three independent ODEs is integrated and hence no stabilization of the unit-length constraint is necessary.
In order to study rotational kinematics of a local frame rigidly attached to an airplane airframe, we start from the kinematic reconstruction equation $\dot{\mathbf{R}}(t)=\mathbf{R}(t)\tilde{\boldsymbol{\omega}}(t)$ that relates the angular velocity $\boldsymbol{\omega}(t)\in\mathcal{R}^3$ of the frame (aircraft's angular velocity) and the time derivative of the frame rotation matrix $\mathbf{R}(t)$ (aircraft attitude matrix). The assignment of the skew symmetric matrix $\tilde{\boldsymbol{\omega}}\in so(3)$ to the vector $\boldsymbol{\omega}\in\mathcal{R}^3$ is an isomorphism of *so*(3) and $\mathcal{R}^3$ [3]. Similarly as Muthe-Kaas approach [4], we seek a solution of kinematic reconstruction equation in the form

$$\mathbf{R}(t)=\mathbf{R}_0\exp(\tilde{\mathbf{u}}(t)), \tag{1}$$

where the Euler-Rodrigues formula [4] provides the closed form of the exponential mapping on *SO*(3)

$$\exp_{SO(3)}(\tilde{\mathbf{u}})=\mathbf{I}_3+\frac{\sin(\|\mathbf{u}\|)}{\|\mathbf{u}\|}\tilde{\mathbf{u}}+\frac{1-\cos(\|\mathbf{u}\|)}{\|\mathbf{u}\|^2}\tilde{\mathbf{u}}^2, \tag{2}$$

and $\mathbf{u}(t)\in\mathcal{R}^3$ is the scaled instantaneous rotation vector, and $\mathbf{R}_0$ is initial attitude. Then, $\tilde{\mathbf{u}}(t)\in so(3)$ satisfies the ODE system in the Lie-algebra

$$\dot{\tilde{\mathbf{u}}}=\mathrm{dexp}^{-1}_{-\tilde{\mathbf{u}}}(\tilde{\boldsymbol{\omega}}(\mathbf{R}(t))),\ \tilde{\mathbf{u}}_0=\mathbf{0}, \tag{3}$$

where the operator $\mathrm{dexp}^{-1}_{-\tilde{\mathbf{u}}}$ admits the series expansion (see [4]), $\mathrm{ad}_{\tilde{\mathbf{u}}}$ is adjoint operator defined by the Lie-bracket and B_j are the Bernoulli numbers.
Unit quaternions form a group which is isomorphic to the symplectic group *Sp*(1) as well as to special unitary group *SU*(2). The unit quaternion group is also isomorphic to the unit sphere in $\mathcal{R}^4$, defined as $\mathcal{S}^3=\left\{\mathrm{q}\in\mathcal{R}^4 \mid \|\mathrm{q}\|=1\right\}$. The rotational motion of a frame is thus described by $\mathrm{q}(t)\in\mathcal{S}^3$, where t denotes the time. The vector space of skew-symplectic quaternions $sp(1)=\{\mathcal{W}\in\mathcal{R}^4 \mid \mathcal{W}+\overline{\mathcal{W}}=(0,\mathbf{0})\}$ as tangent space to $Sp(1)\cong\mathcal{S}^3$ at the group identity, where $\overline{\mathcal{W}}$ is the conjugate of the pure quaternion $\mathcal{W}$. Hence, *sp*(1) is the Lie-algebra of *Sp*(1), which is isomorphic to *so*(3). The Lie-algebra *sp*(1) is the

set of pure imaginary quaternions isomorphic to $\mathcal{R}^3$, so that an element $\mathcal{w} \in sp(1)$ can be assigned to a vector $\mathbf{u} \in \mathcal{R}^3$. For $so(3)$ this assignment was $\tilde{\mathbf{u}}(t) \in so(3)$. In order to ensure that $sp(1)$, $so(3)$, and $\mathcal{R}^3$ are isomorphic as Lie-algebras, the element $\mathcal{w} = (0, 1/2\mathbf{u}) \in sp(1)$ is associated to the vector $\mathbf{u} \in \mathcal{R}^3$. This is very important, and a lapse frequently found in the literature, is to assign $\mathcal{w} = (0, \mathbf{u}) \in sp(1)$. In the proposed algorithm, by following equation (1), we express update for the step in the form

$$\mathrm{q}_{n+1} = \mathrm{q}_n \circ \exp_{S^3}(\mathcal{w}_n) = \mathrm{q}_n \circ \exp_{S^3}((0, 1/2\mathbf{u}_n)), \tag{4}$$

where the closed form of the exponential mapping on S^3 is given by [5]

$$\exp_{S^3}(\mathcal{w}) = \cos(1/2\|\mathbf{u}\|)(1, \mathbf{0}) + \frac{\sin(1/2\|\mathbf{u}\|)}{\|\mathbf{u}\|}(0, \mathbf{u}), \tag{5}$$

and $\mathcal{w}_n$ is element of Lie-algebra $sp(1)$ associated to the incremental rotation vector, and $\mathbf{u}_n \in \mathcal{R}^3$ is the n-th step incremental rotation vector that updates rotation, determined from

$$\dot{\tilde{\mathbf{u}}}_n = \mathrm{dexp}^{-1}_{-\tilde{\mathbf{u}}_n}(\tilde{\boldsymbol{\omega}}(\mathrm{q}(t))), \; \tilde{\mathbf{u}}_{n_0} = \mathbf{0}, \tag{6}$$

where the operator $\mathrm{dexp}^{-1}_{-\tilde{\mathbf{u}}_n}$ is introduced in [4] and $\tilde{\mathbf{u}}_{n_0}$ is initial condition. These equations should be integrated within each integration step together with the equations of aircraft dynamics. Since (6) is an ODE defined in Lie-algebra, that is a vector space, any standard vector-space ODE integrator can be used. Actually, an order of accuracy of the overall algorithm depends only on the accuracy of the ODE integrator that is utilized for solving (6).
For the prime case study a rotation motion of a general 6 DOF aviation airplane is selected.

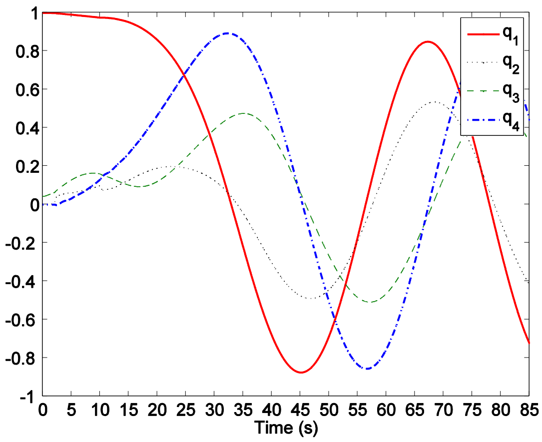

Figure 1: The change of airplane's attitude quaternions as time functions.

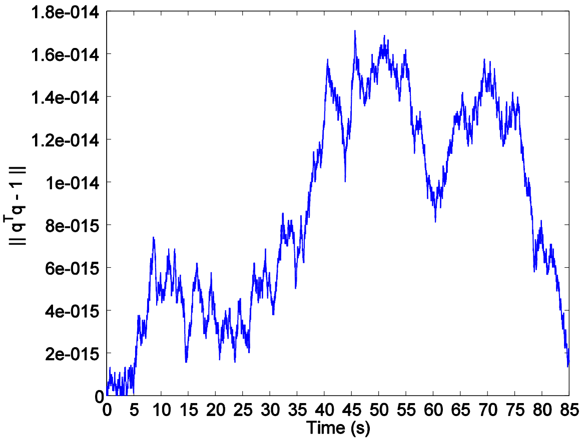

Figure 2: Error in the norm of quaternion unit-length constraint equation.

By inspecting integral curves of the aircraft's rotation (see Fig. 1), it is visible that all obtained results are smooth functions without any discontinuities. Moreover, although obtained directly by integration of 3 ODE in minimal form, geometry of the rotational unit-quaternion is preserved within 'machine' precision, without necessity of solving unit-length algebraic equation (this is illustrated by Fig. 2). Hence the formulation represents a singularity-free approach to the attitude dynamics.

References

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