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A Nilpotent Algebra Approach to Lagrangian Multibody Dynamics

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Abstract

The traditional approach to Lagrangian mechanics involves the evaluation of real-valued scalar functions. Within the Lagrangian framework, one is compelled to construct the requisite equations of motion of a mechanical system by first considering the kinematics of the system and then by evaluating the total system kinetic energy T and potential energy U in order to arrive at the Lagrangian $\mathscr{L}$ of the system. Invariably, the process requires one to find the partial derivatives of $\mathscr{L}$ with respect to (1) the generalized coordinates that describe the system state and (2) time. Furthermore, expressions for U may themselves require many differentiation evaluations with respect to the system configuration. This process can become non-tractable if a large number of coordinates are involved. In the presence of constraints, the situation is further complicated since derivative information of the constraints may be needed for both the development of the constrained equations of motion and their numerical integration. By extensions of the Lagrangian to the commutative Hessian algebra developed herein, it is the goal of this paper to show that the derivation of equations of motion for general constrained dynamical systems can be transformed from an analytical to a purely numerical process.

Schemes for numerical simulation of dynamical systems typically rely on formulations wherein the system topology is determined before actual implementation. This is partly due to the fact that derivatives play a central role in relating the kinematics of a system to its dynamics. Practical numerical implementation of the equations of motion of a constrained dynamical system essentially relies on three different forms of differentiation, which include (1) analytical differentiation by hand, (2) symbolic differentiation, or (3) automatic differentiation. The first option is feasible if all the necessary derivatives can be obtained by hand prior to implementation. However, the process must be repeated with changing system topology and is error prone for complex systems. Symbolic differentiation utilizes a computer algebra system to accomplish the same feat in a more automated fashion, and as a result is less error prone. Both of these forms of differentiation lead to algorithms that rely on concrete analytical expressions. In contrast, automatic differentiation obtains derivatives of functions numerically. Common automatic differentiation methods apply a judicious use of the chain rule to evaluate the derivative of a given function by source code transformation or operator overloading [1, 2]. This assumes that the function of interest is suitably differentiable and can be implemented in a particular programming language. These methods fall into the classification of forward or reverse mode techniques, where forward and reverse are used to indicate the direction traversed by the chain rule. Software implementation of these techniques is widely available [3]. Another popular form of automatic differentiaton includes the complex-step derivative [4, 5, 6], which has been used to numerically compute first and second-order derivatives. However, this approach is based on the finite difference approximation and is subject to truncation errors.

Yet another approach to automatic differentiation involves the use of the so-called dual number algebra. The dual numbers, which can be traced back to the work of Clifford [7], have been shown to produce exact first derivatives of real-valued functions by simply extending the function to the dual number algebra. In mechanics, dual numbers have been primarily used in kinematics analysis; e.g., see [8]. A dual number extension to second derivatives has recently been developed by [9], which was successfully applied to a Navier-Stokes solver. In what follows, a superset of the dual numbers called the Taylor numbers is constructed. It is shown that the Taylor algebra $\mathbb{T}$ can be used to produce derivatives to arbitrary order of continuous real-valued multivariate functions. The Jacobian and Hessian numbers are constructed from truncated Taylor numbers and used in the description of Lagrangian mechanics. Furthermore, it is demonstrated how the algebras can be used to numerically evaluate the elements needed to formulate general constrained equations of motion for dynamical systems. Finally, the extension of the Lagrangian framework to the Hessian algebra is demonstrated for the problem of a flexible multibody system in a central gravitational field.

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