# The Contact Problem in Lagrangian Systems subject to Bilateral and Unilateral constraints with sliding Coulomb's Friction

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**Abstract**

This work concerns the existence and uniqueness of the acceleration and Lagrange multipliers for Lagrangian systems subject to sliding Coulomb's friction with bilateral and unilateral constraints. Focus is put on providing sufficient conditions for singularities like Painlevé paradoxes to be avoided. Explicit criteria, in the form of upper bounds on the friction coefficients, are given so as to preserve the well posedness of the frictional problem.

## 1 Problem formulation

The sliding friction problem consists in determining the acceleration $\ddot{q}$ and the contact forces $\lambda$ of a mechanical system given its state $(q,\dot{q})$ and all other efforts $F(q,\dot{q},t)$ such that the following dynamics is satisfied,

$$\begin{cases} M(q)\ddot{q}+F(q,\dot{q},t) &= \nabla h_{\mathrm{n},b}(q)\lambda_{\mathrm{n},b}+\nabla h_{\mathrm{n},u}(q)\lambda_{\mathrm{n},u}+H_{\mathrm{t},b}(q)\lambda_{\mathrm{t},b}+H_{\mathrm{t},u}(q)\lambda_{\mathrm{t},u} \\ 0\le h_{\mathrm{n},u}(q)\perp\lambda_{\mathrm{n},u}\ge 0 \\ h_{\mathrm{n},b}(q)=0 \\ \lambda_{\mathrm{t},i}=-\mu_i\,\mathrm{sgn}(v_{t,i})|\lambda_{\mathrm{n},i}|. \end{cases} \tag{1}$$

We consider $m_u$ unilateral (inequality) constraints $h_{\mathrm{n},u}(q)\in\mathbb{R}^{m_u}$. The matrix $\nabla h_{n,u}(q)$ collects on each column the gradient for each unilateral constraint, and thus corresponds to the transpose of the Jacobian of $h_{n,u}(q)$. The matrix $H_{\mathrm{t},u}(q)$ maps local tangent frames at the contact points to the generalized coordinates, $v_{t,i}=H_{\mathrm{t},u,i}(q)^T\dot{q}$. The vector of Lagrange multipliers $\lambda_{\mathrm{n},u}\in\mathbb{R}^{m_u}$ corresponds to the normal part of the contact force, its tangential counterpart $\lambda_{\mathrm{t},u}$ is driven by the sliding (single valued) Coulomb friction law. We consider also $m_b$ bilateral (equality) constraints $h_{\mathrm{n},b}(q)\in\mathbb{R}^{m_b}$.

**Frictionless case** Under the assumption of a non-singular mass matrix and linearly independent constraints, the frictionless problem is well-posed. It boils down to the following Linear Complementarity Problem (LCP),

$$0\le\lambda_{\mathrm{n},u}\perp A_c(q)\lambda_{\mathrm{n},u}+w_c(q,\dot{q},t)\ge 0, \tag{2}$$

where $A_c(q)=A_{nu}(q)-A_{nbnu}(q)^TA_{nb}(q)^{-1}A_{nbnu}(q)$ is the Schur complement of $A_{nb}(q)$ in

$$\begin{pmatrix}\nabla h_{\mathrm{n},b}(q)^T\\ \nabla h_{\mathrm{n},u}(q)^T\end{pmatrix}M(q)^{-1}(\nabla h_{\mathrm{n},b}(q)\;\nabla h_{\mathrm{n},u}(q))=\begin{pmatrix}A_{nb}(q) & A_{nbnu}(q)\\ A_{nbnu}(q) & A_{nu}(q)\end{pmatrix},$$

and is thus symmetric positive-definite. Therefore the LCP in (2) has a unique solution for any input vector $w_c(q,\dot{q},t)$ [3].

**Sliding friction case** In the sliding friction case all contacts are closed, frictional and with a non-zero sliding velocity. To handle this case, we convert problem (1) into an LCP with a certain matrix $M^{\mu}_{ub}(q)$. This matrix is then split as $M^{\mu}_{ub}(q)=M^{0}_{ub}(q)+\Theta_\mu(q)$ where $M^{0}_{ub}(q)$ is a P-matrix whose blocks appear in the frictionless problem and $\left\|\Theta_\mu(q)\right\|$ tends to zero as the friction $\mu_i$ goes to zero for each contact. Then using a theorem of [2] which states that a small perturbation of a P-matrix remains a P-matrix, we derive sufficient conditions on the friction coefficients for the matrix $M^{\mu}_{ub}(q)$ to remain a P-matrix.

To convert (1) into an LCP the first step is to introduce slack variables $\lambda^+=\frac{|\lambda_{\mathrm{n},b}|+\lambda_{\mathrm{n},b}}{2}$ and $\lambda^-=\frac{|\lambda_{\mathrm{n},b}|-\lambda_{\mathrm{n},b}}{2}$ which cast the piecewise linearity of the absolute value into a complementarity formalism [1]. The equation for the contact forces then becomes a Mixed LCP (MLCP) of the form

$$\begin{cases} (A_{nb}-A_{tb}[\mu_b\xi_b])\lambda^+ - (A_{nb}+A_{tb}[\mu_b\xi_b])\lambda^- + (A_{nbnu}-A_{nbtu}[\mu_u\xi_u])\lambda_{nu} + w_1 = 0 \\ 0 \leq (A_{nunb}-A_{nutb}[\mu_b\xi_b])\lambda^+ - (A_{nunb}-A_{nutb}[\mu_b\xi_b])\lambda^- + (A_{nu}-A_{tu}[\mu_u\xi_u])\lambda_{nu} + w_2 \perp \lambda_{nu} \geq 0 \\ 0 \leq \lambda^+ \perp \lambda^- \geq 0, \end{cases} \tag{3}$$

where the notation $[\mu\xi]$ refers to the diagonal matrix with entries $\mu_i \mathrm{sgn}(v_{t,i})$. An important observation is that if $\max_{1\leq i\leq m_b}\mu_{b,i} < \mu^b_{\max}(q) \triangleq \frac{\sigma_{min}(A_{nb}(q))}{\sigma_{max}(A_{tb}(q))}$, then the matrix $A_{nb}-A_{tb}[\mu_b\xi_b]$ is positive-definite and the MLCP in (3) boils down to the LCP

$$0 \leq \begin{pmatrix} \lambda^- \\ \lambda_{nu} \end{pmatrix} \perp \underbrace{\begin{pmatrix} A^{-1}\bar{A} & -A^{-1}B \\ CA^{-1}\bar{A}-\bar{C} & D-CA^{-1}B \end{pmatrix}}_{\triangleq M^{\mu}_{ub}(q)} \begin{pmatrix} \lambda^- \\ \lambda_{nu} \end{pmatrix} + \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \geq 0, \tag{4}$$

where $\sigma_{min,max}()$ denote the minimum and maximum singular values of a matrix, $A \triangleq A_{nb}-A_{tb}[\mu_b\xi_b]$, $\bar{A} \triangleq A_{nb}+A_{tb}[\mu_b\xi_b]$, $B \triangleq A_{nbnu}-A_{nbtu}[\mu_u\xi_u]$, $C \triangleq A_{nunb}-A_{nutb}[\mu_b\xi_b]$, $\bar{C} \triangleq A_{nunb}+A_{nutb}[\mu_b\xi_b]$, $D \triangleq A_{nu}-A_{tu}[\mu_u\xi_u]$, $z_1 \triangleq -A^{-1}w_1$ and $z_2 \triangleq w_2 + Cz_1$. Introducing the Taylor series expansion of the matrix $A^{-1}$ is key to decoupling $M^{\mu}_{ub}(q)$ into a frictionless part $M^0_{ub}(q)$ and a frictional part $\Theta_\mu(q)$. Let $K_\mu = \sum_{i=1}^{\infty}(A_{nb}^{-1}A_{tb}[\mu_b\xi_b])^i$, then the blocks of $M^{\mu}_{ub}(q)$ may be decoupled as follows,

$$\begin{aligned} A^{-1}\bar{A} &= I + 2K_\mu \\ -A^{-1}B &= -A_{nb}^{-1}A_{nbnu} + (I+K_\mu)A_{nb}^{-1}A_{nbtu}[\mu_u\xi_u] - K_\mu A_{nb}^{-1}A_{nbnu} \\ CA^{-1}\bar{A}-\bar{C} &= 0 - 2A_{nutb}[\mu_b\xi_b] + CK_\mu \\ D-CA^{-1}B &= A_{nu} - A^T_{nbnu}A_{nb}^{-1}A_{nbnu} - A_{tu}[\mu_u\xi_u] + A_{nutb}[\mu_b\xi_b]A_{nb}^{-1}A_{nbnu} \\ &\quad + C((I+K_\mu)A_{nb}^{-1}A_{nbtu}[\mu_u\xi_u] - K_\mu A_{nb}^{-1}A_{nbnu}). \end{aligned}$$

Hence the frictionless part $M^0_{ub}(q) \triangleq \begin{pmatrix} I & -A_{nb}(q)^{-1}A_{nbnu}(q) \\ 0 & A_c(q) \end{pmatrix}$ is retrieved in the decoupling and is observed to be a P-matrix since $A_c(q)$ is positive-definite. The remaining terms are collected in the matrix $\Theta_\mu(q) \triangleq M^{\mu}_{ub}(q) - M^0_{ub}(q)$, which is readily verified to be of first order in $\mu$, that is $\|\Theta_\mu(q)\|$ tends to zero as $\max_i(\mu_{b,i},\mu_{u,i})$ tends to zero.
We are now in a position to use Theorem 2.8 in [2] regarding small perturbations of P-matrices, to expose a sufficient condition for the existence and uniqueness of a solution to problem (1). Defining for an arbitrary matrix $P$ the quantity $\beta_2(P) \triangleq \max_{d\in[0,1]^n}\|(I-[d]+[d]P)^{-1}[d]\|_2$ as in [2], the following can be stated.

**Proposition 1.** *Let $(q,\dot{q})$, $F(q,\dot{q},t)$ be given and $v_{t,i}\neq 0$ for all i. Suppose that $M(q)$ is positive-definite, that all constraints are independent, that the bilateral friction coefficients satisfy $\max_{1\leq i\leq m_b}\mu_{b,i} < \mu^b_{\max}(q) \triangleq \frac{\sigma_{min}(A_{nb}(q))}{\sigma_{max}(A_{tb}(q))}$, and that $\|\Theta_\mu(q)\|_2 < \frac{1}{\beta_2(M^0_{ub}(q))}$. Then there exists a unique solution $(\ddot{q},\lambda_{nb},\lambda_{nu})$ to the mixed sliding friction problem* (1).

## References


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