ECCOMAS Thematic Conference on Multibody Dynamics
June 29 - July 2, 2015, Barcelona, Catalonia, Spain

A weak formulation for the numerical solution of the equations of motion of multibody systems subjected to scleronomic constraints

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Abstract
This work refers to a weak formulation of the equations of motion for a class of mechanical systems, providing a reliable basis for an efficient numerical determination of their dynamics. The position of these systems is specified by a finite number of generalized coordinates, $q^1,\ldots,q^n$, at any time t [1]. Their motion can then be viewed as motion of a point, say p, along a curve $\gamma=\gamma(t)$ in an n-dimensional configuration manifold M. Moreover, the tangent vector $\underline{v}=d\gamma/dt$ to this curve belongs to an n-dimensional vector space T_pM, the tangent space at p [2]. At every point p, one can define another vector space, the dual or cotangent space to T_pM, denoted by T_p^*M. Also, for any vector $\underline{u}$ in T_pM, a covector $\underset{\sim}{u}^*$ may be identified in T_p^*M by the dual product

$$\underset{\sim}{u}^*(\underline{w})\equiv\langle\underline{u},\underline{w}\rangle,\quad \forall\underline{w}\in T_pM\,, \tag{1}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product of vector space T_pM. In this way, to each basis $\{\underline{e}_i\}$ (with $i=1,\ldots,n$) of T_pM, a dual basis $\{\underset{\sim}{e}^i\}$ can be established for T_p^*M by employing the condition

$$\underset{\sim}{e}^i(\underline{e}_j)=\delta^i_j\,. \tag{2}$$

In addition, the class of systems examined is subjected to a set of k motion constraints with form

$$A(q)\underline{v}=\underline{0}\,, \tag{3}$$

where $\underline{v}$ is a vector in T_pM and $A=[a^R_i]$ is a known $k\times n$ matrix. Considering each of the resulting scalar equations separately, these constraints can be set in the dual product form

$$\dot{\psi}^R(q,\underline{v})\equiv(\underset{\sim}{a}^R(q))(\underline{v})=0\,,\ (R=1,\ldots,k), \tag{4}$$

where $\underset{\sim}{a}^R(q)$ is the R-th row of $A(q)$. When the constraint is holonomic, this can be integrated to

$$\phi^R(q)=0\,.$$

Based on the above, the equations of motion of the class of systems examined can be put in the form

$$\underset{\sim}{h}^*_M=\underset{\sim}{h}^*_C \tag{5}$$

on the configuration manifold M [3], with

$$\underset{\sim}{h}^*_M=[(g_{ij}v^j)^{\cdot}-\Lambda^m_{\ell i}g_{mj}v^jv^\ell-f_i]\underset{\sim}{e}^i \quad\text{and}\quad \underset{\sim}{h}^*_C=\sum\nolimits_{R=1}^{k}[(\overline{m}_{RR}\dot{\lambda}^R)^{\cdot}+\overline{c}_{RR}\dot{\lambda}^R+\overline{k}_{RR}\lambda^R-\overline{f}_R]a^R_i\underset{\sim}{e}^i\,, \tag{6}$$

where the usual convention on repeated indices applies to all indices, except R. The quantities λ^R are appropriate Lagrange multipliers, while the quantities g_{ij} and Λ^ℓ_{ij} represent the components of the metric and the connection on manifold M. The coefficients $\overline{m}_{RR}$, $\overline{c}_{RR}$, $\overline{k}_{RR}$ and $\overline{f}_R$ represent an equivalent mass, damping, stiffness and forcing quantity, obtained through a projection along a special direction on T_pM, specified by the action of the R-th constraint [3]. Finally, for each holonomic or nonholonomic constraint, Eq. (5) is complemented by an ordinary differential equation with form

$$(\overline{m}_{RR}\dot{\phi}^R)^{\cdot}+\overline{c}_{RR}\dot{\phi}^R+\overline{k}_{RR}\phi^R=0 \quad\text{or}\quad (\overline{m}_{RR}\dot{\psi}^R)^{\cdot}+\overline{c}_{RR}\dot{\psi}^R=0\,, \tag{7}$$

respectively. Taking into account the set of equations of motion (5), one can first write

$$\int_{t_1}^{t_2}(\underset{\sim}{h}^*_M-\underset{\sim}{h}^*_C)(\underline{w})dt=0,\quad \forall\underline{w}\in T_pM\,. \tag{8}$$

Also, for each holonomic constraint, as expressed by Eq. (7a), the following relation is satisfied

$$\int_{t_1}^{t_2}[(\overline{m}_{RR}\dot{\phi}^R)^{\cdot}+\overline{c}_{RR}\dot{\phi}^R+\overline{k}_{RR}\phi^R]\delta\lambda^R dt=0\,, \tag{9}$$

for an arbitrary multiplier $\delta\lambda^R$, while a nonholonomic constraint equation can be treated in a similar manner. Moreover, in a weak formulation, it is advantageous to consider the position, velocity and momentum variables as independent quantities [4]. For this, two new velocity fields, $\underline{\upsilon}$ and $\underline{\mu}$, are introduced, which are eventually forced to become identical to the true velocity fields $\underline{v}$ and $\underline{\dot{\lambda}}$. To achieve this task, Eq. (8) should be augmented by the terms

$$\int_{t_1}^{t_2}[\pi_i(\delta\upsilon^i - \delta v^i) + \delta\pi_i(\upsilon^i - v^i)]dt \quad \text{and} \quad \int_{t_1}^{t_2}[\sigma_R(\delta\mu^R - \delta\dot{\lambda}^R) + \delta\sigma_R(\mu^R - \dot{\lambda}^R)]dt, \tag{10}$$

where the quantities π_i and σ_R represent new sets of Lagrange multipliers. The quantities $\delta\pi_i$ and $\delta\sigma_R$ represent components of covectors belonging to the same cotangent space as the covectors with components π_i and σ_R, respectively, but other than that have no relation to them.

Finally, appending Eqs. (9) and (10) to Eq. (8) yields the weak form of the equations of motion for the class of systems examined as a three field set of equations. This form is convenient for performing an appropriate numerical discretization, leading to improvements in existing schemes [5, 6]. In particular, for the purposes of the present work, this form was put within the framework of an augmented Lagrangian formulation. The success of this formulation is demonstrated by the efficient solution obtained for challenging benchmark problems [7]. Results from such an example are shown in Fig. 1.

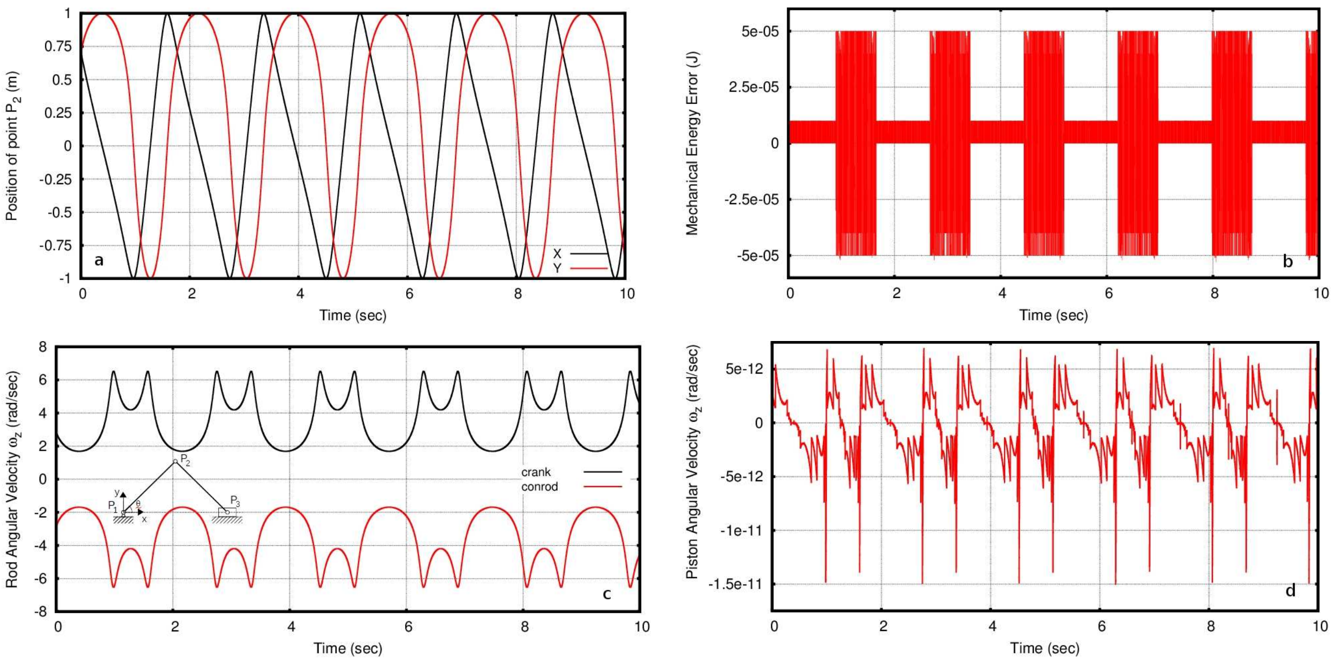

Figure 1: Numerical results for a planar slider-crank mechanism (a) position of point P_2, (b) mechanical energy error, (c) angular velocity of rods, (d) angular velocity of moving piston.

References

[1] A.A. Shabana. Dynamics of Multibody Systems, third ed. Cambridge University Press, New York, 2005.

[2] A.M. Bloch. Nonholonomic Mechanics and Control. Springer-Verlag New York Inc., New York, 2003.

[3] S. Natsiavas, E. Paraskevopoulos. A set of ordinary differential equations of motion for constrained mechanical systems. Nonlinear Dynamics (DOI: 10.1007/s11071-014-1783-5).

[4] K. Rektorys. Variational Methods in Mathematics, Science and Engineering, D. Reidel Publishing Company, Dordrecht, 1977.

[5] M. Geradin, A. Cardona. Flexible Multibody Dynamics. John Wiley & Sons, New York, 2001.

[6] O.A. Bauchau. Flexible Multibody Dynamics. Springer Science+Business Media B.V., London, 2011.

[7] B. Dasguptaa, T.S. Mruthyunjayab. The Stewart platform manipulator: a review. Mechanism and Machine Theory, Vol. 35, No. 1, pp. 15–40, 2000.