# Relationship Between the Natural and Relative Coordinates Through the DeNOC Matrices 

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#### Abstract

In dynamics of the multibody systems, the selection of coordinates is important. Different types of coordinate systems are available for dynamic formulation and have their own advantages and disadvantages. The dynamic formulation of multibody systems based on natural and joint coordinates was done using velocity transformation in [1]. The equations of motion can be formulated easily in large number of natural coordinates (absolute accelerations), however the drawback is the large number of mixed differential-algebraic equations. The numerical solution of these equations is computationally inefficient. On the other hand, the dynamic formulation using the relative coordinates is cumbersome, while they are computationally efficient. Further, the extra effort is required for the computation of the absolute positions, velocities, and accelerations of the multibody systems. A systematic method to derive the minimal set of equations of motion was presented in [2]. The equation of motion was written first in dependent coordinates, then velocity transformation matrix was used to derive the minimal set of equations of motion.

Note that an efficient recursive dynamic formulation, using the DeNOC matrices was developed in [3]. This tempted us to develop a similar set of the DeNOC matrices relating the time-derivatives of the natural coordinates and those of the relative coordinates to obtain a set of required independent equations of motion. The dynamic formulation of two link serial manipulatoris is presented in this paper. A relationship between the natural and relative coordinates was developed at velocity level. We believe that this yields in minimal set of differential equations of motion and provide an efficient numerical solution. In the full paper, we shall develop a transformation matrix for the four-bar mechanism and 3-RRR parallel manipulator and compare the results in terms of efficiency and accuracy.


## Methodology

The two link serial manipulator is shown in Fig. 1. The generalized equation of motion using the Lagrange formulation can be written in terms of natural coordinates as

$$
\begin{equation*}
\mathbf{I}_{\mathrm{n}} \dot{\mathbf{v}}+\mathbf{J}^{\mathrm{T}} \lambda=\emptyset \tag{1}
\end{equation*}
$$

Where $\left(\mathbf{I}_{\mathbf{n}}\right)$ is the generalized inertia matrix (GIM) in natural coordinates, $\mathbf{J}$ is the constraint Jacobian matrix, $\lambda$ is the Lagrange multiplier, $\varnothing$ is the generalized force vector, and $\mathbf{v}$ is the vector of linear velocities of the joints. The $4 \times 4 \mathrm{GIM}$ is given by

$$
\mathbf{I}_{\mathbf{n}}=\left[\begin{array}{cccc}
\frac{\left(\mathrm{m}_{1}+\mathrm{m}_{2}\right)}{3} & 0 & \frac{\mathrm{~m}_{2}}{6} & 0  \tag{2}\\
0 & \frac{\left(\mathrm{~m}_{1}+\mathrm{m}_{2}\right)}{3} & 0 & \frac{\mathrm{~m}_{2}}{6} \\
\frac{\mathrm{~m}_{2}}{6} & 0 & \frac{\mathrm{~m}_{2}}{3} & 0 \\
0 & \frac{\mathrm{~m}_{2}}{6} & 0 & \frac{\mathrm{~m}_{2}}{3}
\end{array}\right]
$$

The kinematic relation between the natural and relative coordinates is given below:

$$
\mathbf{v}_{1}=\left[\begin{array}{l}
\dot{x}_{1}  \tag{3}\\
\dot{y}_{1}
\end{array}\right]=\dot{\mathbf{c}}_{1}+\boldsymbol{\omega}_{1} \times \mathbf{r}_{1}=\left[\begin{array}{ll}
-\mathbf{r}_{1} \times \mathbf{1} & \mathbf{1}
\end{array}\right]\left[\begin{array}{c}
\boldsymbol{\omega}_{1} \\
\dot{\mathbf{c}}_{1}
\end{array}\right]
$$

$$
\mathbf{v}_{2}=\left[\begin{array}{l}
\dot{x}_{2}  \tag{4}\\
\dot{\mathrm{y}}_{2}
\end{array}\right]=\dot{\mathbf{c}}_{2}+\boldsymbol{\omega}_{2} \times \mathbf{r}_{2}=\left[\begin{array}{ll}
-\mathbf{r}_{2} \times \mathbf{1} & \mathbf{1}
\end{array}\right]\left[\begin{array}{l}
\boldsymbol{\omega}_{2} \\
\dot{\mathbf{c}}_{2}
\end{array}\right]
$$

Then,

$$
\left[\begin{array}{l}
\mathbf{v}_{1}  \tag{5}\\
\mathbf{v}_{2}
\end{array}\right]=\left[\begin{array}{cccc}
-\mathbf{r}_{1} \times \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & -\mathbf{r}_{2} \times \mathbf{1} & \mathbf{1}
\end{array}\right]\left[\begin{array}{c}
\boldsymbol{\omega}_{1} \\
\dot{\mathbf{c}}_{1} \\
\boldsymbol{\omega}_{2} \\
\dot{\mathbf{c}}_{2}
\end{array}\right]
$$



Figure 1: Two link serial manipulator.

The equation (5) can be written in compact form as given below:

$$
\begin{equation*}
\mathbf{v}=\mathbf{R} \mathbf{t} \tag{6}
\end{equation*}
$$

From [3], we have twist vector $\mathbf{t}=\mathbf{N}_{\mathbf{l}} \mathbf{N}_{\mathrm{d}} \dot{\boldsymbol{\theta}}$, therefore the above can be written as $\mathbf{v}=\mathbf{R} \mathbf{N}_{\mathbf{l}} \mathbf{N}_{\mathrm{d}} \dot{\boldsymbol{\theta}}$, and in compact form $\mathbf{v}=\mathbf{T} \dot{\boldsymbol{\theta}}$, where $\mathbf{T}=\mathbf{R} \mathbf{N}_{\mathrm{l}} \mathbf{N}_{\mathrm{d}}$, which gives $\dot{\mathbf{v}}=\dot{\mathbf{T}} \dot{\boldsymbol{\theta}}+\mathbf{T} \ddot{\boldsymbol{\theta}}$. Put the value of the $\dot{\mathbf{v}}$ in eq. (1), and pre-multiply the transpose of $\mathbf{T}$ to the both side of eq. (1). This operation transforms the equation of motion in relative coordinates. The generalized form of equation of motion for the serial chain manipulator in relative coordinates is given below:

$$
\begin{equation*}
\mathbf{I}_{\mathrm{r}} \ddot{\boldsymbol{\theta}}+\mathbf{C} \dot{\boldsymbol{\theta}}=\boldsymbol{\tau} \tag{7}
\end{equation*}
$$

Where ( $\mathbf{I}_{\mathrm{r}}$ ) is the GIM in relative coordinates, $\mathbf{C}$ is the matix of convetive inertia and $\boldsymbol{\tau}$ is the generalized force vector. The transformed expression in relative coordinates for the GIM is given below:

$$
\boldsymbol{I}_{r}=\left[\begin{array}{cc}
\frac{\left(m_{1} a_{1}^{2}+m_{2} a_{2}^{2}\right)}{3}+m_{2} a_{1}^{2}+m_{2} a_{1} a_{2} \cos \theta_{2} & \frac{m_{2} a_{2}^{2}}{3}+\frac{1}{2} m_{2} a_{1} a_{2} \cos \theta_{2}  \tag{8}\\
\frac{m_{2} a_{2}^{2}}{3}+\frac{1}{2} m_{2} a_{1} a_{2} \cos \theta_{2} & \frac{m_{2} a_{2}^{2}}{3}
\end{array}\right]
$$

In the full paper, numerical results will be presented.

## References

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