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A mixed shooting and harmonic balance method for mechanical systems with dry friction or other local nonlinearities

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Introduction

In this paper we present a mixed shooting – harmonic balance method for large linear mechanical systems with local nonlinearities. The standard harmonic balance method (HBM) approximates the periodic solution in frequency domain and is very popular as it is well suited for large systems with many states. Local nonlinearities cannot be evaluated directly in the frequency domain. The standard HBM performs an inverse Fourier transform, then calculates the nonlinear force in time domain and subsequently the Fourier coefficients of the nonlinear force. The disadvantage of the HBM is, that strong nonlinearities are poorly represented by a truncated Fourier series. In contrast, the shooting method operates in time-domain and relies on numerical time-simulation. Set-valued force laws such as dry friction or other strong nonlinearities can be dealt with if an appropriate numerical integrator is available. The shooting method, however, becomes infeasible if the system has many states. The proposed mixed shooting–HBM approach combines the best of both worlds.

System description

We consider a Lagrangian system

$$\boldsymbol{M}\ddot{\boldsymbol{q}}(t)+\boldsymbol{C}\dot{\boldsymbol{q}}(t)+\boldsymbol{K}\boldsymbol{q}(t)=\boldsymbol{f}_{\mathrm{ex}}(t)+\boldsymbol{f}_{\mathrm{NL}}(\boldsymbol{q}(t),\dot{\boldsymbol{q}}(t)) \tag{1}$$

where $\boldsymbol{f}_{\mathrm{NL}}$ contains the nonlinear forces and $\boldsymbol{f}_{\mathrm{ex}}(t)=\boldsymbol{f}_{\mathrm{ex}}(t+T)$ is the periodic forcing. We assume that the system consists of three subsystems with $\boldsymbol{q}=\begin{pmatrix}\boldsymbol{q}_1^{\mathrm{T}} & \boldsymbol{q}_2^{\mathrm{T}} & \boldsymbol{q}_3^{\mathrm{T}}\end{pmatrix}^{\mathrm{T}}$, that the nonlinear forces only act on subsystem 1, and that the system matrices $\boldsymbol{M}$, $\boldsymbol{C}$ and $\boldsymbol{K}$ have the following structure

$$\boldsymbol{M}=\begin{pmatrix}\boldsymbol{M}_{11} & \boldsymbol{M}_{12} & \boldsymbol{0}\\ \boldsymbol{M}_{21} & \boldsymbol{M}_{22} & \boldsymbol{M}_{23}\\ \boldsymbol{0} & \boldsymbol{M}_{32} & \boldsymbol{M}_{33}\end{pmatrix}, \qquad \boldsymbol{f}_{\mathrm{NL}}(\boldsymbol{q},\dot{\boldsymbol{q}})=\begin{pmatrix}\boldsymbol{f}_{\mathrm{NL1}}(\boldsymbol{q}_1,\dot{\boldsymbol{q}}_1)\\ \boldsymbol{0}\\ \boldsymbol{0}\end{pmatrix}. \tag{2}$$

Subsystem 1 is subjected to a nonlinear force, which only depends on its own positions and velocities, and is connected to subsystem 3 through subsystem 2, e.g. the three DOF oscillator shown in Figure 1.

Mixed shooting–HBM approach

For subsystem 2 and 3 we use a harmonic balance approach and impose (as a numerical approximation) perfect constraints on the system which force the response to be harmonic of the form

$$\boldsymbol{q}_2(t)=\hat{\boldsymbol{q}}_2^0+\sum_{k=1}^{n}\hat{\boldsymbol{q}}_2^{ck}\cos k\omega t+\hat{\boldsymbol{q}}_2^{sk}\sin k\omega t=\boldsymbol{V}(t)^{\mathrm{T}}\hat{\boldsymbol{q}}_2, \qquad \boldsymbol{q}_3(t)=\boldsymbol{V}(t)^{\mathrm{T}}\hat{\boldsymbol{q}}_3, \tag{3}$$

where $\omega=\frac{2\pi}{T}$. The motion $\boldsymbol{q}_1(t)$ of subsystem 1 is not constrained to be harmonic. The equations of motion of subsystem 2 and 3 can therefore be expressed in frequency domain as

$$\begin{aligned}\boldsymbol{H}_{21}\hat{\boldsymbol{q}}_1+\boldsymbol{H}_{22}\hat{\boldsymbol{q}}_2+\boldsymbol{H}_{23}\hat{\boldsymbol{q}}_3&=\hat{\boldsymbol{f}}_{\mathrm{ex2}},\\ \boldsymbol{H}_{32}\hat{\boldsymbol{q}}_2+\boldsymbol{H}_{33}\hat{\boldsymbol{q}}_3&=\hat{\boldsymbol{f}}_{\mathrm{ex3}},\end{aligned} \tag{4}$$

where $\hat{\boldsymbol{q}}_1$ are the Fourier coefficients of $\boldsymbol{q}_1(t)$ and $\boldsymbol{H}_{ij}$ are the matrices

$$\boldsymbol{H}_{ij}=diag(\boldsymbol{J}_{ij,0},\boldsymbol{J}_{ij,1},\dots\boldsymbol{J}_{ij,n}) \quad \text{with} \quad \boldsymbol{J}_{ij,k}=\begin{pmatrix}-\boldsymbol{M}_{ij}(k\omega)^2+\boldsymbol{K}_{ij} & \boldsymbol{C}_{ij}k\omega\\ -\boldsymbol{C}_{ij}k\omega & -\boldsymbol{M}_{ij}(k\omega)^2+\boldsymbol{K}_{ij}\end{pmatrix} \tag{5}$$

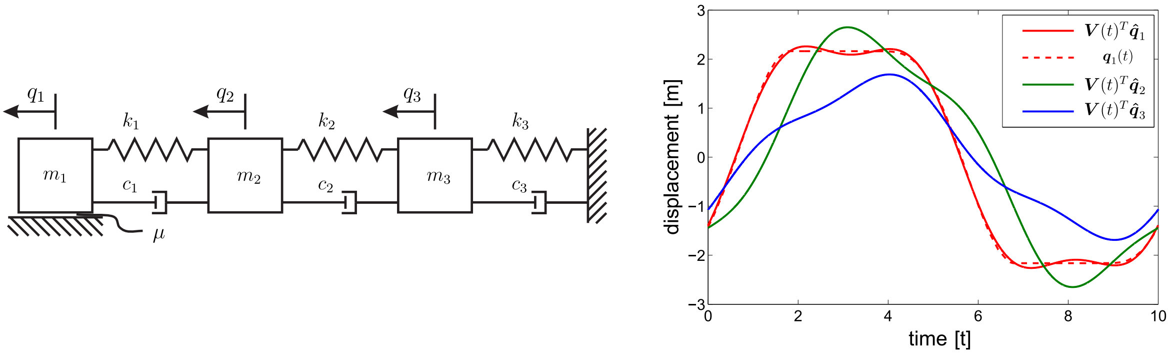

Figure 1: Mixed shooting-HBM for a three DOF oscillator.

The Fourier coefficients $\hat{\boldsymbol{q}}_3$ can be eliminated from the equations of motion in frequency domain

$$\boldsymbol{H}_{21}\hat{\boldsymbol{q}}_1 + (\boldsymbol{H}_{22} - \boldsymbol{H}_{23}\boldsymbol{H}_{33}^{-1}\boldsymbol{H}_{32})\hat{\boldsymbol{q}}_2 = \hat{\boldsymbol{f}}_{\mathrm{ex2}} - \boldsymbol{H}_{23}\boldsymbol{H}_{33}^{-1}\hat{\boldsymbol{f}}_{\mathrm{ex3}}. \tag{6}$$

The equations of motion of subsystem 1 are nonlinear and are simulated in time-domain. For known $\hat{\boldsymbol{q}}_2$ one can calculate its time-domain representation $\boldsymbol{q}_2(t)$ and solve the differential equation

$$\boldsymbol{M}_{11}\ddot{\boldsymbol{q}}_1(t) + \boldsymbol{C}_{11}\dot{\boldsymbol{q}}_1(t) + \boldsymbol{K}_{11}\boldsymbol{q}_1(t) = -(\boldsymbol{M}_{12}\ddot{\boldsymbol{q}}_2(t) + \boldsymbol{C}_{12}\dot{\boldsymbol{q}}_2(t) + \boldsymbol{K}_{12}\boldsymbol{q}_2(t)) + \boldsymbol{f}_{\mathrm{ex1}}(t) + \boldsymbol{f}_{\mathrm{NL1}}(\boldsymbol{q}_1(t), \dot{\boldsymbol{q}}_1(t)) \tag{7}$$

using numerical integration techniques.
A periodic solution of the system can be represented by the trajectory $\boldsymbol{q}_1(t)$ on the interval $0 \le t \le T$ and by the Fourier coefficients $\hat{\boldsymbol{q}}_2$ and $\hat{\boldsymbol{q}}_3$. However, $\hat{\boldsymbol{q}}_3$ can be eliminated. The initial condition $\boldsymbol{q}_1(0)$ and $\dot{\boldsymbol{q}}_1(0)$ together with $\boldsymbol{q}_2(t) = \boldsymbol{V}(t)^{\mathrm{T}}\hat{\boldsymbol{q}}_2$ suffice to construct $\boldsymbol{q}_1(t)$ over one period. The vector of unknowns $\boldsymbol{x} = \begin{pmatrix}\hat{\boldsymbol{q}}_2^{\mathrm{T}} & \boldsymbol{q}_1(0)^{\mathrm{T}} & \dot{\boldsymbol{q}}_1(0)^{\mathrm{T}}\end{pmatrix}$ therefore fully represents a periodic solution of the system. Similar to a shooting method, we require for subsystem 1 the periodicity conditions $\boldsymbol{q}_1(T) - \boldsymbol{q}_1(0)$ and $\dot{\boldsymbol{q}}_1(T) - \dot{\boldsymbol{q}}_1(0)$, where the state at $t = T$ is obtained through numerical integration of (7). The periodicity conditions of subsystems 2 and 3 are given in frequency domain by (6). Hence, we seek a periodic solution by finding a zero of the nonlinear function

$$\boldsymbol{f}(\boldsymbol{x}) = \begin{pmatrix} \boldsymbol{H}_{21}\hat{\boldsymbol{q}}_1 + (\boldsymbol{H}_{22} - \boldsymbol{H}_{23}\boldsymbol{H}_{33}^{-1}\boldsymbol{H}_{32})\hat{\boldsymbol{q}}_2 - \hat{\boldsymbol{f}}_{\mathrm{ex2}} + \boldsymbol{H}_{23}\boldsymbol{H}_{33}^{-1}\hat{\boldsymbol{f}}_{\mathrm{ex3}} \\ \boldsymbol{q}_1(T) - \boldsymbol{q}_1(0) \\ \dot{\boldsymbol{q}}_1(T) - \dot{\boldsymbol{q}}_1(0) \end{pmatrix}, \tag{8}$$

which can be solved with a Newton-type method.
Alternativly we may iterate in the unknowns $\boldsymbol{x} = \begin{pmatrix}\hat{\boldsymbol{q}}_1^{\mathrm{T}} & \boldsymbol{q}_1(0)^{\mathrm{T}} & \dot{\boldsymbol{q}}_1(0)^{\mathrm{T}}\end{pmatrix}$ by solving $\hat{\boldsymbol{q}}_2(\hat{\boldsymbol{q}}_1)$ from (6), simulating $\boldsymbol{q}_1(t)$ from (7) and requiring that the Fourier coefficients of $\boldsymbol{q}_1(t)$ are equal to $\hat{\boldsymbol{q}}_1$.

Conclusion

We have presented a mixed shooting–HBM method which has a number of advantages. Firstly, in contrast to the standard shooting method, the number of degrees of freedom of subsystem 3 can be very large without any additional numerical costs. Secondly, in contrast to the standard HBM, the motion of subsystem 1 is represented in time-domain which allows for a very good description of the influence of the nonlinear force. For instance, the dry friction force in the example system of Figure 1 causes stick–slip motion in the response of subsystem 1 which is well described by the mixed approach. Lastly, the mixed shooting–HBM method allows to use modern time-stepping methods [1, 2] to deal with friction and unilateral constraints within a harmonic balance approach.

References

[1] Acary, V., and Brogliato, B. *Numerical Methods for Nonsmooth Dynamical Systems; Applications in Mechanics and Electronics*, vol. 35 of *LNACM*. Springer, Berlin, 2008.

[2] Leine, R. I., and Nijmeijer, H. *Dynamics and Bifurcations of Non-Smooth Mechanical Systems*, vol. 18 of *LNACM*. Springer, Berlin, 2004.