Towards a Maximal Monotone Impact Law for Newton's Cradle

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Introduction

In this paper, we present an impact law for Newton's Cradle with 3 balls which is kinematically and kinetically consistent *and* enjoys the maximal monotonicity property. Our aim is to divulge the structure of impact laws in order to be able to formulate maximal monotone impact laws for rigid multi-body systems that do not have the problems of existing impact laws such as kinematic, kinetic, and energetic inconsistency [1]. It is interesting to consider Newton's Cradle because its phenomena cannot be described by the classical Newton's or Poisson's impact laws.
Kinematic consistency means that the impenetrability of unilateral constraints requires the post-impact contact velocities γ_i^+ to be non-negative. In view of numerical integration, an impact-law should guarantee that arbitrary (also kinematically inconsistent) pre-impact contact velocities are mapped to kinematically consistent post-impact contact velocities.
The interest in the maximal monotonicity property stems from stability analysis and control of mechanical systems with unilateral constraints [3]. Since the maximal monotonicity property implies dissipativity it seems to be a physically reasonable property of an impact law.

An Impact Law for the 3-ball Newton's Cradle

In the following, an impact law is presented for the 3-ball Newton's Cradle (see Figure 1). The proposed system consists of three balls of equal mass m with positions $\mathbf{q} = (q_1\ q_2\ q_3)^{\mathrm{T}}$ and velocities $\dot{\mathbf{q}} = \mathbf{u} = (u_1\ u_2\ u_3)^{\mathrm{T}}$. The contact velocities are given by the relative velocities between the balls $\boldsymbol{\gamma} = (\gamma_1\ \gamma_2)^{\mathrm{T}} = (u_2 - u_1\ u_3 - u_2)^{\mathrm{T}}$. The pre- and post-impact velocities are designated by $\mathbf{u}^-$ and $\mathbf{u}^+$ respectively. Analogously, $\boldsymbol{\gamma}^-$ and $\boldsymbol{\gamma}^+$ designate the pre- and post-impact relative velocities.

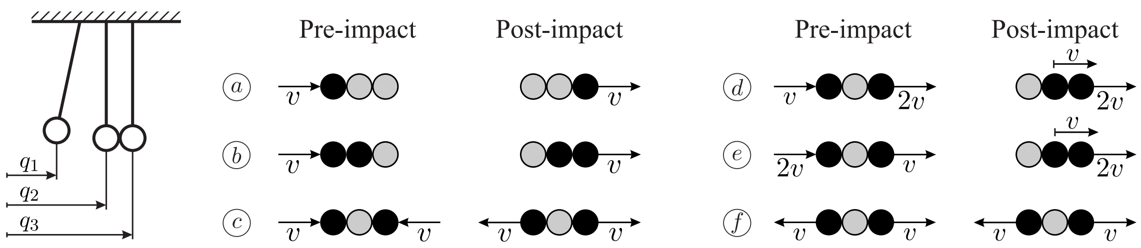


Figure 1: *Left:* Newton's Cradle with 3 balls of mass m. *Middle:* Kinematically consistent pre-impact velocities. *Right:* Kinematically inconsistent pre-impact velocities.

The impact equations of the system can be written in the following matrix form

$$\mathbf{M}(\mathbf{u}^+ - \mathbf{u}^-) = \mathbf{W}\boldsymbol{\Lambda}, \tag{1}$$

$$\boldsymbol{\gamma}^{\pm} = \mathbf{W}^{\mathrm{T}}\mathbf{u}^{\pm}, \tag{2}$$

where $\boldsymbol{\Lambda} = (\Lambda_1\ \Lambda_2)^{\mathrm{T}}$ are the impulsive contact forces during the impact. The impulsive force Λ_1 acts between balls 1 and 2, while Λ_2 occurs between balls 2 and 3. The matrix $\mathbf{W} = \{\mathbf{w}_i\}$ is the matrix of generalized force directions $\mathbf{w}_i^{\mathrm{T}} = \frac{\partial g_i}{\partial \mathbf{q}}$, with $\mathbf{g} = \{g_i\} = (q_2 - q_1\ q_3 - q_2)^{\mathrm{T}}$ being the vector of the gap functions of the contacts between the balls. For the 3-ball Newton's Cradle, the matrices $\mathbf{M}$ and $\mathbf{W}$ are respectively

$$\mathbf{M} = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \quad \text{and} \quad \mathbf{W} = \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}. \tag{3}$$

$$
\begin{array}{ccccc}
S:\boldsymbol{\gamma}^- \mapsto \boldsymbol{\gamma}^+ & & -\boldsymbol{\Lambda} \in \mathcal{H}(\bar{\boldsymbol{\gamma}}) & & Z:\mathbf{u}^- \mapsto \mathbf{u}^+ \\
\text{non-expansive in } \mathbf{G}^{-1} & \Longleftrightarrow & \text{maximal monotone} & \Longleftrightarrow & \text{non-expansive in } \mathbf{M} \\
\|\boldsymbol{\gamma}_A^+ - \boldsymbol{\gamma}_B^+\|_{\mathbf{G}^{-1}} \leq \|\boldsymbol{\gamma}_A^- - \boldsymbol{\gamma}_B^-\|_{\mathbf{G}^{-1}} & & -(\boldsymbol{\Lambda}_A - \boldsymbol{\Lambda}_B)^{\mathrm{T}}(\bar{\boldsymbol{\gamma}}_A - \bar{\boldsymbol{\gamma}}_B) \geq 0 & & \|\mathbf{u}_A^+ - \mathbf{u}_B^+\|_{\mathbf{M}} \leq \|\mathbf{u}_A^- - \mathbf{u}_B^-\|_{\mathbf{M}}
\end{array}
$$

Figure 2: Interrelations of a maximal monotone impact law [2].

The impact equation (1) needs to be complemented by an impact law that has the mathematical structure of a set-valued relationship [2]

$$-\boldsymbol{\Lambda} \in \mathcal{H}(\bar{\boldsymbol{\gamma}}), \tag{4}$$

where $\bar{\boldsymbol{\gamma}} = \frac{1}{2}(\boldsymbol{\gamma}^+ + \boldsymbol{\gamma}^-)$ and $\mathcal{H} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is in general a set-valued operator. By combination of eq. (1) and eq. (2), we obtain

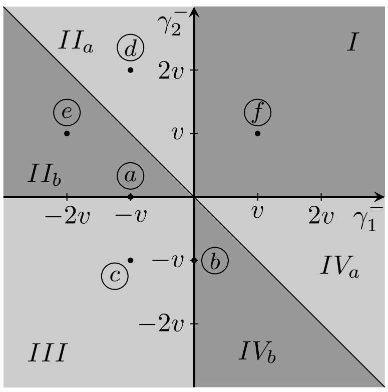
$$\boldsymbol{\gamma}^+ - \boldsymbol{\gamma}^- = \mathbf{G}\boldsymbol{\Lambda}, \quad \text{with } \mathbf{G} := \mathbf{W}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{W}. \tag{5}$$

From [4, 2], it is known that the maximal monotonicity of (4) is equivalent to non-expansivity properties of the impact mapping $\boldsymbol{\gamma}^+ = S(\boldsymbol{\gamma}^-)$ respectively of $\mathbf{u}^+ = Z(\mathbf{u}^-)$. The corresponding relations are shown in Figure 2.

In this work, a piece-wise linear relation is proposed for the impact mapping S such that

$$\boldsymbol{\gamma}^+ = S(\boldsymbol{\gamma}^-) = \mathbf{Q}_i\boldsymbol{\gamma}^-, \tag{6}$$

where the $\mathbf{Q}_i \in \mathbb{R}^{2\times 2}$ are the 2-by-2 matrices that are given in Figure 3. A major result of the paper is that one can rigorously prove that the mapping (6) is non-expansive in the metric $\mathbf{G}^{-1}$. Therefore, it follows from Figure 2 that the impact law (4) is maximal monotone. The impact mapping (6) guarantees kinematic consistency by mapping any pre-impact velocity $\boldsymbol{\gamma}^-$ to the first quadrant (Sector I in Figure 3).



$$
\mathbf{Q}_I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \mathbf{Q}_{III} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix},
$$

$$
\mathbf{Q}_{II_a} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}, \qquad \mathbf{Q}_{IV_b} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix},
$$

$$
\mathbf{Q}_{II_b} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \qquad \mathbf{Q}_{IV_a} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}.
$$

Figure 3: Matrices $\mathbf{Q}_i$ of the impact mapping S for the different cones of the (γ_1^-, γ_2^-)–plane.

Conclusion

A maximal monotone impact law has been formulated which can describe the wave-like phenomena in Newton's cradle and is kinematically, kinetically, and energetically consistent. The proposed impact mapping (6) may allow to reveal further details about the general structure of set-valued impact laws (4).

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