Velocity-based approach in non-linear dynamics of three-dimensional beams with quaternion representation of rotations

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Abstract

Beam elements are important members of flexible multibody systems. Among various existing non-linear beam theories we here study the geometrically exact beam theory as introduced by Antman [1], Reissner [2], and Simo [3]. An efficient numerical formulation of this model for dynamic analysis is still a challenge for researches. Most of the problems reported stem from the properties of three-dimensional rotations that are directly or indirectly incorporated into the solution methods. The non-linearity of three-dimensional rotations requires a special treatment in parameterization, discretization, and update. It is crucial for the overall efficiency of the numerical formulation that in all these procedures a sufficient attention is paid in preserving the properties of rotations after the discretization.

There is a number of possible ways of choosing the suitable representation or parameterization of spatial rotations, see e.g. the textbooks [4] or [5]. We will limit our studies to the quaternion representation of rotations as the applicability of quaternions in three-dimensional beam theories has been recently well recognized [6, 7].

In this work, we present the derivation of dynamic governing equations of the three-dimensional beam in terms of quaternions that follows from the d'Alembert's principle extended with the unit norm constraint of the rotational quaternion. We also present a novel numerical solution approach based on the interpolation of velocities and angular velocities along the length of the beam. The same quantities are taken to be the primary quantities of the time-discretization scheme. Such approach is in sharp contrast to previously presented quaternion-based beam formulations for dynamics, where the displacements and rotational quaternions were the primary unknowns. Despite the advantages of the quaternion-based approach, already reported in literature, this replacement of the primary unknowns has proven to be beneficial for the increase of numerical stability and robustness of the model.

The continuous balance equations of a three-dimensional beam in quaternion notation read

$$\mathbf{n}' + \tilde{\mathbf{n}} - \rho A_r \ddot{\mathbf{r}} = \mathbf{0} \tag{1}$$

$$\left[\mathbf{m}' + \mathbf{r}' \times \mathbf{n} + \tilde{\mathbf{m}} - \widehat{\mathbf{q}} \circ \left(\mathbf{J}_\rho \dot{\boldsymbol{\Omega}}\right) \circ \widehat{\mathbf{q}}^* - \boldsymbol{\omega} \times \left(\widehat{\mathbf{q}} \circ \left(\mathbf{J}_\rho \boldsymbol{\Omega}\right) \circ \widehat{\mathbf{q}}^*\right) - \lambda \widehat{\mathbf{1}}\right] \circ \widehat{\mathbf{q}} = \widehat{\mathbf{0}} \tag{2}$$

$$\widehat{\mathbf{q}} \cdot \widehat{\mathbf{q}} - 1 = 0, \tag{3}$$

where $\circ$ denotes the quaternion product, $\mathbf{n}$ and $\mathbf{m}$ are stress-resultant force and moment vectors, $\tilde{\mathbf{n}}$ and $\tilde{\mathbf{m}}$ are external distributed force and moment vectors per unit of the initial length; $\mathbf{r}$ is the position vector, $\widehat{\mathbf{q}}$ is rotational quaternion, $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$ are angular velocities with respect to fixed and moving basis, respectively, ρ denotes mass per unit of the initial volume, A_r is the area of the cross-section; $\mathbf{J}_\rho$ is the centroidal mass-inertia matrix of the cross-section. Note, that the balance equation (2) differs from the standard angular momentum balance equation, but we can show the equivalence to the standard one. However, while for a continuous system these two equivalent sets of equations would theoretically lead to the same result, in the numerical approach different form of equations could considerably affect the behaviour of the numerical method.

In constructing the time integrator we follow the idea of Brüls et al. [8] and Češarek and Zupan [9] and express the time dependent quantities of the second-order differential equation with the first derivatives of the configuration unknowns:

$$\mathbf{r}^{[n+1]} = \mathbf{r}^{[n]} + h\overline{\mathbf{v}}, \quad \mathbf{v}^{[n+1]} = \frac{\beta-1}{\beta}\mathbf{v}^{[n]} + \frac{1}{\beta}\overline{\mathbf{v}}, \quad \boldsymbol{a}^{[n+1]} = \frac{\gamma-1}{\gamma}\boldsymbol{a}^{[n]} - \frac{1}{\beta\gamma h}\mathbf{v}^{[n]} + \frac{1}{h\gamma\beta}\overline{\mathbf{v}}. \tag{4}$$

Similar approach can be used for rotational degrees of freedom, with an important exception: the relationship between angular velocities and rotational quaternions is non-linear, thus:

$$\widehat{\mathbf{q}}^{[n+1]} = \widehat{\mathbf{q}}^{[n]} \circ \exp\left(\frac{h}{2}\overline{\mathbf{\Omega}}\right), \quad \mathbf{\Omega}^{[n+1]} = \frac{\beta-1}{\beta}\mathbf{\Omega}^{[n]} + \frac{1}{\beta}\overline{\mathbf{\Omega}}, \quad \boldsymbol{\alpha}^{[n+1]} = \frac{\gamma-1}{\gamma}\boldsymbol{\alpha}^{[n]} - \frac{1}{\beta\gamma h}\mathbf{\Omega}^{[n]} + \frac{1}{h\gamma\beta}\overline{\mathbf{\Omega}}. \quad (5)$$

The primary variables of this approach are the average velocity $\overline{\mathbf{v}}$ and the average angular velocity $\overline{\mathbf{\Omega}}$ on the time interval $[t_n, t_{n+1}]$, while $\beta, \gamma \in [0,1]$ are the parameters. The scheme (5) is expressed in terms of quaternions with the exponential map denoting the *quaternion exponential*, defined by infinite power series:

$$\exp(\widehat{\mathbf{x}}) = \sum_{k=1}^{\infty} \frac{\widehat{\mathbf{x}}^k}{1!} = \widehat{\mathbf{1}} + \frac{\widehat{\mathbf{x}}}{1!} + \frac{1}{2!}\widehat{\mathbf{x}} \circ \widehat{\mathbf{x}} + \frac{1}{3!}\widehat{\mathbf{x}} \circ \widehat{\mathbf{x}} \circ \widehat{\mathbf{x}} + ... \quad (6)$$

We base the spatial discretiation on the same to quantities

$$\overline{\mathbf{v}}(x) = \sum_{p=1}^{N} I_p(x)\,\overline{\mathbf{v}}^p, \quad \overline{\mathbf{\Omega}}(x) = \sum_{p=1}^{N} I_p(x)\,\overline{\mathbf{\Omega}}^p, \quad (7)$$

which are the only interpolated unknowns along the lenght of the beam.
After the discretization we need to solve a system of discrete nonlinear algebraic equations at each time step. The present choice of rotational parameters and the primary unknowns results in relatively simple and numerically efficient linearization of equations. Besides that, we are able to update the unknowns consistently which contributes to the long term numerical stability and robustness of the algorithm.

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