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Conceptual and numerical aspects of the mixed variational formulation of geometrically exact beam models

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Abstract

We concern ourselves with geometrically exact beam formulations and their numerical applications. A rigorous approach to derive the governing equations of the beam is to apply the principle of virtual work in the static case or its generalization for dynamics – d'Alembert's principle. An important part of deriving the governing equations is the treatment of the kinematic equations. In the classical approach, the strains are taken as dependent variables and are obtained from the kinematic equations by means of interpolation of the configuration variables which are subsequently differentiated. This leads to formulations with displacement and rotational parameters as primary variables – so called configuration based formulations. In our work, we highlight the benefits (w.r.t. conceptual as well as numerical aspects) of an alternative formulation of mixed type, which is based on the treatment of equilibrium equations, kinematic relations and constitutive laws as separate entities.

Equilibrium problems for beams can likewise be formulated using the strains as independent variables, where the kinematic equations are added to the virtual work principle with the method of Lagrange multipliers, leading to the so called Hu-Washizu functional. This leads to mixed-type formulations where strains and Lagrange multipliers take the role of the primary variables. Examples of such formulations are the strain-based formulation of Zupan and Saje [1] and the intrinsic dynamic formulation presented by Hodges [2]. The governing set of equations obtained from mixed-type variational formulations consist of kinematic equations, equilibrium equations and their boundary conditions, and a special set of equations here denoted as the consistency conditions. The equilibrium equations reveal the physical meaning of the Lagrange multipliers, which are recognized as the cross-sectional stress resultant force f and moment m of the beam, while the consistency conditions impose the requirement that f and m are equal to the constitutive force f^C and moment m^C, respectively, along each point of the beam centerline

$$f^C - f = 0 \ , \ m^C - m = 0 \ . \tag{1}$$

Usually it is assumed that the consistency conditions (1) are exactly satisfied, such that they could be eliminated from the governing equations, which reduces the size of the problem. However, after the problem is discretized, the consistency is not necessarily preserved. From this perspective it is advantageous not to eliminate the consistency conditions from the problem, but to keep them as separate equations. In mixed-type finite element formulations the generalized forces are members of the primary variables. This means that the equilibrium forces f and moments m, which take the role of Lagrange multipliers associated to the preservation of kinematic constraints, are demanded to be equal to the corresponding constitutive quantities computed from the strains, at least in the discrete sense. In such a way the shear locking problem is automatically eliminated from the finite element formulation. Furthermore, the equality of equilibrium forces and moments and the constitutive forces and moments is of utmost importance in materially nonlinear problems, which is demonstrated in [1]. In mixed formulations, where the consistency conditions are members of the system of governing equations, the configuration space consists of kinematic variables (configuration parameters and also, but not necessarily, strains) and generalized forces. Therefore, the configuration space is larger as for the classical formulations where it is spanned solely by displacements and/or rotational parameters. Despite a larger storage space required,

the configuration space in mixed formulation is simpler, which improves the performance (faster convergence, larger load/prescribed displacement steps) of the iterative solvers (usually Newton's method) used to solve the system of nonlinear equations.

The improved performance of the solution method results in faster computational times, which is a highly desirable property of flexible multibody applications. A particularly attractive choice of beam elements for such applications are elements with assumed constant strains [3] or linear interpolation of displacements and spherical linear interpolation of rotations [4, 5]. The strain-based formulation [3] is of mixed-type (SB-mixed formulation). When the strains are eliminated from the set of primary variables by analytical integration of the kinematic equations, it leads to a formulation with consistency conditions, where configuration parameters and generalized forces are the only primary variables (CB-mixed formulation). A further reduction of such formulation involves the extraction of generalized forces from the constitutive equations which are assumed to be satisfied. This leads to classical configuration-based (CB) formulation where only displacement and rotational parameters are the primary unknowns.

In order to demonstrate the performance of the formulations described above, we present the simulation results of a 45° cantilever bend [6] in Table 1. The first observation from Table 1 is that all of the

Table 1: 45° cantilever bend

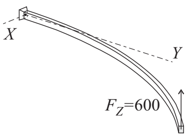

figure	formulation	element d.f.	iter. per inc.	total iter.	converged results r_X	r_Y	r_Z	relative comp. time for 4 inc.
	SB-mixed	24	5	20	15.80	47.23	53.37	0.83
	CB-mixed	18	8	31	15.80	47.23	53.37	0.50
	CB	12	13	52	15.80	47.23	53.37	1.00

SB – strain based, CB – configuration based

formulations converge to the same result. The load was applied in four steps. In this case the mixed formulations require less iterations than the configuration-based formulation to converge, which results in a faster solution of the problem. For example, the relative computational times (relative to the CB formulation) for 4 load increments are presented in the last column. The observed substantial reduction of the computational time of the mixed formulation is not only due to the smaller number of iterations required in each load step, but also due to the fact that in each iteration the structure of the linearized problem (the Jacobian matrix) is simpler as for the classical approach, which enables faster calculation.

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