A Formulation of a Thin Plate Element based on the Absolute Nodal Coordinate Formulation with Artificial Constraints

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Abstract

Plate elements based on the absolute nodal coordinate formulation (ANCF) have been studied for the last decade[1, 2, 3, 4]. In the ANCF, two different approaches have been developed for the definition of the elastic forces. The first method is a so-called continuum mechanics approach. In the another method, a local element coordinate system is employed for the description of the element deformation. However, some literatures pointed out difficulties due to highly nonlinear terms for elastic forces in this method. This paper focuses on the description of the elastic forces by the latter approach. In order to remove the difficulties due to the introduction of the local element coordinate system from the description of the plate deformation, we develop an analytical procedure with artificial constraints based on the Hamiltonian mechanics.

The thin plate element introduced here is shown in Figure 1[2]. In this formulation, we consider the displacements $\boldsymbol{u}$ in the local element coordinate system as the nodal coordinates as well as the global position $\boldsymbol{R}$. For a node i $(i = 1, 2, 3, 4)$, the nodal coordinate vectors of the global position and the local displacement are defined as follows:

$$\boldsymbol{q}^{(i)} = \left(\begin{array}{cccc} \boldsymbol{u}^{(i)} & \dfrac{\partial \boldsymbol{u}^{(i)}}{\partial x} & \dfrac{\partial \boldsymbol{u}^{(i)}}{\partial y} & \dfrac{\partial^2 \boldsymbol{u}^{(i)}}{\partial x \partial y} \end{array} \right)^T, \quad \boldsymbol{Q}^{(i)} = \left(\begin{array}{cccc} \boldsymbol{R}^{(i)} & \dfrac{\partial \boldsymbol{R}^{(i)}}{\partial X} & \dfrac{\partial \boldsymbol{R}^{(i)}}{\partial Y} & \dfrac{\partial^2 \boldsymbol{R}^{(i)}}{\partial X \partial Y} \end{array} \right)^T, \tag{1}$$

Introducing a block diagonal matrix $[\mathbf{T}]$ consisting of the rotation matrix $[\mathbf{T}_e]$ (direction cosine) from the global coordinate system to the element coordinate system, the relation between the global and element coordinate systems can be expressed by

$$\boldsymbol{q}^{(i)} + \boldsymbol{q}_0 - [\mathbf{T}](\boldsymbol{Q}^{(i)} - \boldsymbol{Q}_0) = \mathbf{0}\,, \tag{2}$$

$$\boldsymbol{q}_0 = \left(\begin{array}{cccc} \boldsymbol{r}_0 & \dfrac{\partial \boldsymbol{r}_0}{\partial x} & \dfrac{\partial \boldsymbol{r}_0}{\partial y} & \dfrac{\partial^2 \boldsymbol{r}_0}{\partial x \partial y} \end{array} \right)^T, \quad \boldsymbol{Q}_0 = \left(\begin{array}{cccc} \boldsymbol{R}_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} \right)^T,$$

$$[\mathbf{T}] = \mathrm{diag}\{\mathbf{T}_e,\ \mathbf{T}_e,\ \mathbf{T}_e,\ \mathbf{T}_e\}\,,$$

where $\boldsymbol{r}_0$ is the position vector in the element coordinate system for each element defined by $\boldsymbol{r}_0 = (x \ \ y \ \ 0)^T$ and $\boldsymbol{R}_0$ is the position vector of the origin of each element in the global coordinate system defined by $\boldsymbol{R}_0 = (X_0 \ \ Y_0 \ \ Z_0)^T$. The specific expressions of $[\mathbf{T}_e]$ can be obtained as functions of the global coordinates $\boldsymbol{Q}$ and unit vectors in the element coordinate system $\boldsymbol{e}_i\,(i = x, y, z)$. Due to the existence of the algebraic relation defined by (2), the set of nodal coordinates in (1) involves redundant degrees of freedom obviously. In order to solve this problem, the relation defined by (2) is introduced as the constraints $\boldsymbol{G}$. It yields the Lagrangian functional L with the constraints as follows:

$$L = T(\dot{\boldsymbol{Q}}) - U(\boldsymbol{q}) + \boldsymbol{\lambda}^T \boldsymbol{G}(\boldsymbol{Q}, \boldsymbol{q}, \boldsymbol{e}_i)\,, \tag{3}$$

where $\boldsymbol{\lambda}$ denotes the Lagrange multiplier vector. The Hamiltonian functional H can be derived by the Legendre transform with (3). It gives

$$H = T + U - \boldsymbol{\lambda}^T \boldsymbol{G} + \boldsymbol{\mu}^T \boldsymbol{\phi}\,, \tag{4}$$

where the functionals T and U are the kinetic and potential energies, respectively. The another Lagrange multiplier vector $\boldsymbol{\mu}$ is introduced against the absences of the canonical momenta $\boldsymbol{\pi}$ corresponding to the quantities $\boldsymbol{q}$, $\boldsymbol{e}_i$ and $\boldsymbol{\lambda}$, that is constraints $\boldsymbol{\phi} = \boldsymbol{\pi} = \mathbf{0}$, are also considered by Lagranges's method of undetermined multipliers. These Lagrange multipliers $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ can be derived by the consistency conditions for the constraints $\boldsymbol{\phi} = \mathbf{0}$, that is $d\boldsymbol{\phi}/dt = \mathbf{0}$, $d^2\boldsymbol{\phi}/dt^2 = \mathbf{0}$, $\cdots$. These time derivatives can be calculated by using the Poisson bracket $\{f, g\}$, where f and g are arbitrary functions (see [5]). More specifically, the algebraic equations derived by $d\boldsymbol{\phi}/dt = \{\boldsymbol{\phi}, H\} = \mathbf{0}$ give the Lagrange multipliers $\boldsymbol{\lambda}$. Consequently, this formulation gives the differential-algebraic equations (DAEs) constituting a set of

$$\dot{\boldsymbol{Q}} = \frac{\partial T}{\partial \boldsymbol{P}}, \qquad \dot{\boldsymbol{P}} = -\frac{\partial \boldsymbol{G}^T}{\partial \boldsymbol{Q}}\boldsymbol{\lambda}, \qquad \{\boldsymbol{\phi}, H\} = \mathbf{0}, \tag{5}$$

where $\boldsymbol{P}$ denotes the canonical momentum vector corresponding to the global position vector $\boldsymbol{Q}$. The present equations in (5) can be classified as DAEs with index one. In the case of the ANCF, the algebraic equations $\{\boldsymbol{\phi}, H\} = \mathbf{0}$ can be solved analytically. Therefore, iteration procedures are not required in the calculation for the Lagrange multipliers $\boldsymbol{\lambda}$.

Figure 2 shows the numerical example for the thin rectangular plate. One side of the plate is simply supported to the ground and it is subjected to a gravity force. The plate structure is discretized by the Hermitian rectangle element[2] using 25^2 elements. We implemented the simulations with the classical strain models such as Cauchy strain and von Kármán strain under the assumptions of Kirchoff-Love plate. A remarkable benefit of the present method in (5) is the computational efficiency.

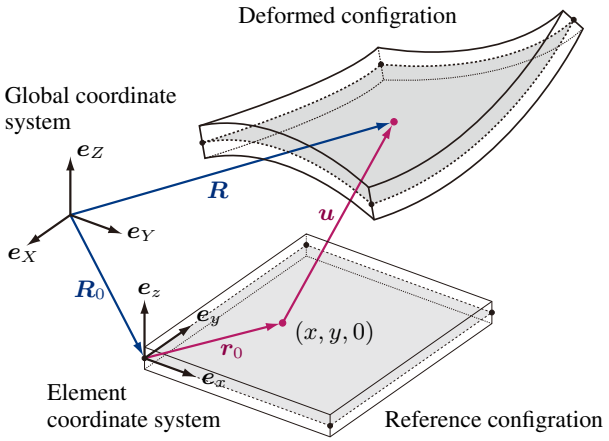

Figure 1: Analytical model of a 4-noded thin plate element.

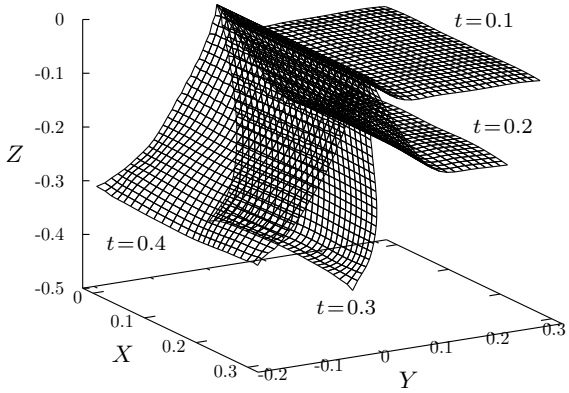

Figure 2: Simulation result of a free falling of a thin plate (Cauchy strain model).

References

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