PLENARY LECTURE

TREFFTZ METHODS: PRESENT AND FUTURE

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Abstract

Trefftz originally proposed his method in 1926, but it has been in the last decades, especially since Jirousek and his collaborators originated hybrid-Trefftz (HT) finite element (FE) model that it has become a highly efficient computational tool for the solution of difficult boundary value problems. These methods have received important contributions for their development from many different fields of application; among them: potential problems, plane elasticity, plate bending (thin, thick, post-buckling), heat conduction, advective–diffusive transport. More recently, Herrera interpreted Trefftz method as a domain decomposition method (DDM) and proposed a unified theory based on it. Herrera's version of Trefftz method supplies a general perspective, which yields a systematic framework for very broad classes of numerical methods for partial differential equations. Then, the approaches can be classified into two wide categories: *direct* -or Trefftz-Jirousek- and *indirect* -or Trefftz-Herrera- *methods*. Herrera's unified theory expands considerably Trefftz method scope and yields new avenues of thought for numerical methods of partial differential equations. Although several important results, such as a very general formulation of DDMs, together with the Localized Adjoint Method (LAM) and broad families of numerical approximations, the future developments are even more promising. This lecture will be devoted to explain the present state and the expected future of these methodologies.

1. INTRODUCTION

Trefftz introduced his method in a paper published in 1926 [1]. However, the origins of the hybrid-Trefftz (HT) finite element (FE) model are only around twenty five years old [2, 3]. Since then it has become a highly efficient computational tool for the solution of difficult boundary value problems [4, 5] with an increasing popularity among researchers and practitioners [6-12]. In parallel and to a large extent independently, a general and elegant theory of domain decomposition methods (DDM) has been developed by Herrera and coworkers [13-21]. This, throughout its different stages of development, has been known by a variety of names; mainly, localized adjoint method (LAM), Trefftz-Herrera method and unified theory of DDM. This is a general formulation, which subsumes and generalizes many other approaches. In particular, it is the natural framework for Trefftz methods and several aspects of that theory have been recognized as fundamental by some of the most conspicuous researchers of these methodologies [4, 22 and 23]. Thus, based on it, a unified theory of Trefftz method has been developed [19, 21], which expands its scope. Indeed, any partial differential equation or system of such equations, independently of its type, can be dealt with. Furthermore, the general problem treated is one with prescribed jumps in the internal boundary and the differential operators coefficients may have jump discontinuities. In addition, such theory can be applied

for constructing discretization procedures, as well as for incorporating parallel computing resources in numerical models of continuous systems of science and engineering.

2. UNIFIED THEORY OF TREFFTZ METHODS

The basic idea of the unified theory Trefftz Method is explained next [19]. Consider a boundary-value problem –or initial boundary-value problem- for a partial differential equation, or system of such equations, formulated in a domain Ω. Then, given a partition $\Pi \equiv \{\Omega_1, ..., \Omega_E\}$ of such a domain, the general problem of domain decomposition methods consists in establishing procedures, which permit solving the 'global' problem defined in Ω, by solving exclusively 'local' problems defined in each one of the subdomains of the partition Ω_i, $i = 1, ..., E$. The unified theory of Trefftz methods considers procedures for gathering information -about the global solution of the problem- in the internal boundary of the partition $\sum$ –i.e., that which separates the subdomains from each other-. In the general method of the unified theory, a target of information on $\sum$ –the '*sought-information*'- is defined beforehand, such that it is sufficient for defining well-posed local problems, which the global solution must satisfy in each one of the subdomains of the partition Ω_i, $i = 1, ..., E$. Then a search is directed to obtaining the *sought-information*. Two very broad categories of procedures for achieving this goal are identified by the theory: 'direct' (or Trefftz-Jirousek) and 'indirect' (or Trefftz-Herrera) methods.

In the usual interpretation of direct methods, they are seen as techniques for building the global solution by putting together, just as 'bricks', the local solutions. In the unified theory, however, a slightly more sophisticated point of view is adopted, since the local solutions of the differential operator are used to establish compatibility relations that the *sought-information* must fulfill. These relations give rise to the global system of equations, from which the *sought-information* is obtained.

In Trefftz-Herrera methods, on the other hand, a system of weighting functions of a special kind, with the property of yielding the *sought-information* in the internal boundary, exclusively, is developed and applied. The idea of constructing such test functions stems from the observation that, in the method of weighted residuals, the information about the exact solution that the approximate one contains, depends on the system of weighting functions, which is applied [24]. And, in order to fabricate the special test functions, it is necessary to have a procedure for analyzing such dependence. In the theory of indirect domain decomposition methods, the basic ingredient of such analysis are Green-Herrera formulas. These formulas were originated by Herrera, in 1985 [14,15], and apply when both, trial and test functions, are fully discontinuous; something that cannot be done when the standard theory of distributions is used. They have already played a fundamental role in establishing the theoretical foundations of a method that is extensively used in water resources studies; the Eulerian Lagrangian Localized Adjoint Method [25] (ELLAM). Using them, necessary and sufficient conditions that the test functions must fulfill, in order to yield the *sought-information* exclusively, are established. Also, a characterization of the *sought-information* in terms of a variational principle (or weak characterization) is supplied, which holds when test functions of the special kind described above, are applied. This principle constitutes a very general, although abstract, formulation of

indirect methods [6]. In addition, techniques to fabricate the special kind of test functions are developed in the theory.

Reference [20] supplies an updated version of indirect domain decomposition methods, where the theory and its applications to problems in several dimensions are discussed with considerable detail. In addition, a plenary lecture of the 14th International Conference on Domain Decomposition Methods was devoted to the indirect method of domain decomposition [26] and further applications can be found in its Proceedings [27]. In the applications that have been done thus far, many advantages of collocation methods derived from this approach have been exhibited [27,28]. Among others, generally the structure of the global matrix is simpler and, if desired, a significant reduction of the number of degrees of freedom associated with each node can be achieved. A common feature of 'direct', when formulated as in the unified theory, and 'indirect' methods is that in both, the information about the solution is obtained at the internal boundary exclusively and, consequently, the interpolation functions used to approximate the sought solution are also applied in the internal boundary only. Such information can be extended to the interior of the subdomains of the partition by a procedure, which is implied by the theory, referred as 'optimal interpolation' [29].

3. SCOPE AND APPLICATIONS

The generality of the methodologies presented in this article is great, since they are applicable to any partial differential equation or system of such equations, which is linear, independently of its type. The coefficients of the operators can also be discontinuous across the internal boundary Σ and the problems treated include prescribed discontinuities across Σ. To illustrate the wide theory applicability the following cases are here mentioned:

i. The general elliptic equation of second order;
ii. The equations of equilibrium of linear elasticity;
iii. The biharmonic equation;
iv. Stokes problem;
v. The heat equation and parabolic equations, in general;
vi. The wave equation and the 3-D equations of elastodynamics.

Trefftz-Jirousek approach has been extensively applied to symmetric problems with constant coefficients, mainly. This is due to the fact that authors working with Trefftz-Jirousek method use T-complete function systems of analytical solutions exclusively. Qin [5] has done a very comprehensive survey of such applications. The fields covered by them include potential problems, plane and 3-D elastostatics, thin plates, moderately thick (Reissner-Mindlin) and thick plates, plate bending, transient heat, elastoplasticity and dynamics of plate bending.

Applications of Trefftz-Herrera approach and of the recently developed Trefftz unified theory, on the other hand, do not require the use of analytical T-complete systems. In particular, this method is applicable, and has been applied to problems with variable coefficients. Also, the procedures can be used to develop discretization procedures and for parallel processing partial differential equations. Thus far, it has been applied to elliptic equations [5], the biharmonic equation [28] and parabolic equations [30].

Variational and weak formulations are essential, and play a central role in the formulation of the basic equations of Trefftz methods. During the past decades much work on them has been

done. Basic for Trefftz-Jirousek approach are those of Piltner [31], Jirousek [32] and Jirousek and Zielinski [33]. Very general variational and weak formulations for the unified theory were given by Herrera and his collaborators [14,15, 17, 25] and their relations with those of Trefftz-Jirousek approach were recently established [34].

4. NOTATIONS

In this paper D is a linear space; elements of D are functions. Functional-valued operators of the form $P: D \to D^*$, which are linear, will be considered. Here, D^* is the linear space whose elements are the real-valued linear functionals defined on D. Given $f \in D^*$, its value $f(v)$ at any $v \in D$ is denoted by $\langle f, v \rangle$, while $Pu \in D^*$ will be the value of $P: D \to D^*$ at any $u \in D$. Therefore, given $P: D \to D^*$, the expression $\langle Pu, v \rangle$ defines a unique bilinear functional on $D \times D$ and this establishes a one-to-one correspondence between the class of linear functional-valued operators considered and such bilinear functionals. Given $f \in D^*$ and $u \in D$, the equation $Pu = f$ is an equality between linear functionals and, as such, it is tantamount to

$$\langle Pu, w \rangle = \langle f, w \rangle \quad \forall w \in D \tag{4.1}$$

The notations $\Omega \subset R^n$ and $\partial\Omega$ will be used for a domain of the Euclidean space of dimension n and its boundary, respectively. In the first part of our discussion, n can be any natural number, but starting with Section 4, n is taken to be equal to 2. Let $\Pi \equiv \{\Omega_1, ..., \Omega_E\}$ be a partition of Ω. Given such a partition, the boundaries of the subdomains are $\partial\Omega_i$, $i = 1, ..., E$. Clearly, $\partial\Omega \subset \bigcup_{i=1}^{E} \partial\Omega_i$ and the "internal boundary" Σ of Ω is defined to be the closed complement of $\partial\Omega$ relative to $\bigcup_{i=1}^{E} \partial\Omega_i$. It is assumed that for each $i = 1, ..., E$, there is a linear space $D(\Omega_i)$, whose elements are functions defined in Ω_i. Then the linear space D is defined to be

$$D \equiv D(\Omega) \equiv D(\Omega_1) \oplus ... \oplus D(\Omega_E) \tag{4.2}$$

Possible choices for $D(\Omega_i)$ are the Sobolev spaces $H^s(\Omega_i)$, $i = 1, ..., E$. For the case of elliptic equations of second order that will be considered, it is convenient to take $s \geq 2$. In fact, when the space D is defined by Eq.(4.2), a function $u \in D$ is a finite sequence of functions $u \equiv (u_1, ..., u_E)$ such that $u_i \in D(\Omega_i)$, $i = 1, ..., E$. It is assumed that the trace of every $u_i \in D(\Omega_i)$ is defined at every point of $\partial\Omega_i$, except for, possibly, a set of measure zero. At any point of Σ, again with the possible exception of a set of measure zero, there is defined a unit normal vector $\underline{n}$ and the manifold Σ is oriented in this manner, taking as positive the side where $\underline{n}$ points to. Given a function $u \in D$, $u \equiv (u_1, ..., u_E)$, two traces are defined at every point of Σ, which are denoted by u_+ *and* u_-, respectively. Since $u_+ \neq u_-$, it is useful to define the '*jump*' and the '*average*' of any function $u \in D$ by

$$[u] = u_+ - u_- \text{ and } \dot{u} = (u_+ + u_-)/2 \tag{4.3}$$

respectively. Clearly, the definition of the jump of a function is dependent on the orientation of Σ; however, the expressions that will be handled in this paper are invariant with respect to such orientation. In some previous works, for simplicity, we have written $\mathcal{L}u = f_\Omega$, in Ω, to mean

$$\mathcal{L}u = f_\Omega, \text{ at each } \Omega_i,\ i = 1,...,E \tag{4.4}$$

For greater clarity, in the present paper we will be more explicit and write directly, Eq.(4.4), since generally $w\mathcal{L}u$ is not defined on Σ, when $u \in D$ and $w \in D$. Similarly, we also write $\sum_{i=1}^{E}\int_{\Omega_i} w\mathcal{L}u dx$ instead of $\int_\Omega w\mathcal{L}u dx$. Assume a tensor-valued function $\underline{\underline{a}}$ is defined in Ω, then it can be shown that

$$\sum_{i=1}^{E}\int_{\partial\Omega_i} w\underline{a}_n \cdot \nabla u dx = \int_{\partial\Omega} w\underline{a}_n \cdot \nabla u dx - \int_\Sigma \left[w\underline{a}_n \cdot \nabla u\right] dx \tag{4.5}$$

The unit normal vector is taken pointing outwards both, on $\partial\Omega$ and $\partial\Omega_i$, $i = 1,...,E$. On Σ, it is taken as explained before.

5.- INDIRECT FORMULATION OF DOMAIN DECOMPOSITION

The following presentation is based on [20]. To start, let $\mathcal{L}$ be a differential operator and $\mathcal{L}^*$ its formal adjoint. Then, there exists a vector-valued bilinear function, $\underline{\mathcal{D}}(u,w)$, which satisfies

$$w\mathcal{L}u - u\mathcal{L}^* w \equiv \nabla\bullet\underline{\mathcal{D}}(u,w); \tag{5.1}$$

Additional bilinear functionals $\mathcal{B}(u,w)$, $\mathcal{C}(w,u)$, $\mathcal{J}(u,w)$ and $\mathcal{K}(w,u)$, defined point-wise, are introduced, such that

$$\underline{\mathcal{D}}(u,w)\bullet\underline{n} \equiv \mathcal{B}(u,w) - \mathcal{C}^*(u,w) \quad \text{on } \partial\Omega \tag{5.2}$$

and

$$-[\underline{\mathcal{D}}(u,w)]\bullet\underline{n} \equiv \mathcal{J}(u,w) - \mathcal{K}^*(u,w) \quad \text{on } \Sigma \tag{5.3}$$

For the case when the coefficients of the differential operators are continuous, Herrera [17, 25] has given very general formulas for bilinear functionals $\mathcal{J}$ and $\mathcal{K}$, which fulfill Eq.(5.3); they are:

$$\mathcal{J}(u,w) \equiv -\underline{\mathcal{D}}([u],\dot{w})\bullet\underline{n}, \text{ and } \mathcal{K}^*(u,w) \equiv \underline{\mathcal{D}}(\dot{u},[w])\bullet\underline{n} \tag{5.4}$$

By virtue of the generalized Gauss theorem it is seen that Eqs. (5.1) to (5.3), together, imply the following Green formula:

$$\int_\Omega w\mathcal{L}u dx - \int_{\partial\Omega}\mathcal{B}(u,w)dx - \int_\Sigma \mathcal{J}(u,w)dx \equiv \int_\Omega u\mathcal{L}^* w dx - \int_{\partial\Omega}\mathcal{C}^*(u,w)dx - \int_\Sigma \mathcal{K}^*(u,w)dx \tag{5.5}$$

Defining the bilinear functionals

$$\langle Pu, w\rangle \equiv \int_\Omega w\mathcal{L}u dx; \qquad \langle Q^* u, w\rangle \equiv \int_\Omega u\mathcal{L}^* w dx; \tag{5.6}$$

$$\langle Bu, w\rangle \equiv \int_{\partial\Omega}\mathcal{B}(u,w)dx; \qquad \langle C^* u, w\rangle \equiv \int_{\partial\Omega}\mathcal{C}^*(u,w)dx; \tag{5.7}$$

$$\langle Ju, w\rangle \equiv \int_\Sigma \mathcal{J}(u,w)dx; \qquad \langle K^* u, w\rangle \equiv \int_\Sigma \mathcal{K}^*(u,w)dx; \tag{5.8}$$

Eq. (5.5) can be written as an identity between bilinear functionals:

$$P - B - J \equiv Q^* - C^* - K^*; \tag{5.9}$$

Given functions $u_\Omega \in D_1$, $u_\partial \in D_1$ and $u_\Sigma \in D_1$, the general boundary value problem with prescribed jumps (*BVPJ*) to be considered is

$$\mathcal{L}u = f_\Omega, in\ \Omega_i, i = 1,..,E \tag{5.10}$$

$$\mathcal{B}(u,w) = \langle g_\partial, w\rangle, on\ \partial\Omega \tag{5.11}$$

and

$$\mathcal{J}(u,w) = \langle j_\Sigma, w\rangle, on\ \Sigma \tag{5.12}$$

which hold for $\forall w \in D_2$. Here

$$f_\Omega \equiv \mathcal{L}u_\Omega, in\ \Omega_i, i = 1,..,E \tag{5.13}$$

and the linear functionals $g_\partial \in D_2^*$ and $j_\Sigma \in D_2^*$, are defined by

$$\langle g_\partial, w\rangle \equiv \mathcal{B}(u_\partial, w)\ and\ \langle j_\Sigma, w\rangle \equiv \mathcal{J}(u_\Sigma, w), \forall w \in D_2 \tag{5.14}$$

Introducing the notation

$$f \equiv Pu_\Omega,\ g \equiv Bu_\partial \text{ and } j \equiv Ju_\Sigma, \tag{5.15}$$

where f, g and $j \in D_2^*$, a weak formulation of the *BVPJ* can be expressed as an equality between linear functionals

$$(P - B - J)u = f - g - j \tag{5.16}$$

which can be written more explicitly as

$$\langle (P - B - J)u, w\rangle = \langle f - g - j, w\rangle \quad \forall w \in D \tag{5.17}$$

This is referred as the *'variational formulation in terms of the data'* and, by virtue of the Green formula of Eq. (5.9), it is equivalent to the following *'variational formulation in terms of the complementary information'*

$$\langle (Q^* - C^* - K^*)u, w\rangle = \langle f - g - j, w\rangle \quad \forall w \in D \tag{5.18}$$

As mentioned in the Introduction, the general strategy of the indirect approach to domain decomposition methods consists in defining a target of information on the internal boundary, referred to as the "sought information", and developing procedures for gathering it. To this end, a weak formulation characterizing the *sought information*, which constitutes the basis of the indirect approach to domain decomposition, is derived next. The bilinear functional K is decomposed by means of two bilinear functionals, S and R, which fulfill

$$K \equiv S + R \tag{5.19}$$

and, when $u \in \hat{D}_1$ is a solution of the BVPJ, '*the sought information*' is defined to be S^*u. In particular, a function $\hat{u} \in D$ is said to '*contain the sought information*' when $S^*\hat{u} = S^*u$. The test functions that yield the sought information, exclusively, constitute a linear subspace, $N \equiv N_Q \cap N_C \cap N_R \subset D_2$. For any function $\hat{u} \in D_1$, the Green-Herrera formula of Eq. (5.9) implies

$$-\langle S^*\hat{u}, w\rangle = \langle (P - B - J)\hat{u}, w\rangle \quad \forall w \in N \tag{5.20}$$

The basic variational principle, on which the indirect approach is based, is:

Theorem 3.1.- *Let $\mathcal{E} \subset N \equiv N_Q \cap N_C \cap N_R$ be a TH-complete system for S^*. Then, a necessary and sufficient condition for $\hat{u} \in \hat{D}_1$ to contain the sought information, is that*

$$-\langle S^*\hat{u}, w\rangle = \langle f - g - j, w\rangle; \quad \forall w \in \mathcal{E} \tag{5.21}$$

This theorem was introduced in [4].

Eq. (5.21), which characterizes variationally the *sought information* on Σ, constitutes a general abstract formulation of Trefftz-Herrera Domain Decomposition. It only yields information on the internal boundary, but in the theory it is assumed that such information, when complemented with Eqs. (5.10) to (5.12) is sufficient for defining well posed problems in each one of the subdomains of the partition, separately. However, knowledge of S^*u does not supply any information about the solution in the interior of the subdomains Ω_i and the natural way of extending this information into the interior of those subdomains is by means of *'optimal interpolation'*. By this, we mean actually solving the above mentioned well-posed local boundary-value problems in each one of the subdomains Ω_i *(i = 1,..., E)*.

6. ELLIPTIC EQUATIONS

In [20], collocation methods for the general non-symmetric elliptic equation of second order were derived. However, here for simplicity only the symmetric case will be treated Thus, the BVPJ to be discussed is

$$\mathcal{L}u \equiv -\nabla\bullet\left(\underline{a}\bullet\nabla u\right)+cu = f_\Omega, \text{ on } \Omega_i,\ i=1,...,E \tag{6.1}$$

with boundary and jump conditions given by

$$u = u_\partial \equiv \mathrm{g}_\partial \ on\ \partial\Omega \tag{6.2}$$

$$[u]=[u_\Sigma]\equiv j_\Sigma^0 \ \ and\ \ [\underline{a}_n\bullet\nabla u]=[\underline{a}_n\bullet\nabla u_\Sigma]\equiv j_\Sigma^1 \ \ on\ \ \Sigma \tag{6.3}$$

The trial and test functions will be taken from the same linear space D, defined in Section 4. Above $c\geq 0$, while $u_\partial\ and\ u_\Sigma \in D$ are auxiliary functions which may be used for prescribing the boundary and jump conditions, respectively. In addition, it is assumed that the boundary conditions and jump conditions are compatible. By this we mean that there exists a function $u_{\partial\Sigma}\in D$ such that

$$u_{\partial\Sigma}=u_\partial, \text{ on } \partial\Omega,\ [u_{\partial\Sigma}]=[u_\Sigma] \text{ and } \left[\underline{a}_n\cdot\nabla u_{\partial\Sigma}\right]=\left[\underline{a}_n\cdot\nabla u_\Sigma\right] \text{ on } \Sigma \tag{6.4}$$

Application of Eq. (3.1) yields

$$\underline{\mathcal{D}}(u,w)=\underline{\underline{a}}\bullet(u\nabla w - w\nabla u) \tag{6.5}$$

for this case. Also, for the symmetric differential operator of Eq. (6.1) it is possible to choose $\mathcal{B}(u,w)\equiv\mathcal{C}(u,w)$ and $\mathcal{J}(u,w)\equiv\mathcal{K}(u,w)$, as we will do. Then, suitable choices are

$$\mathcal{B}(u,w)\equiv u\underline{a}_n\bullet\nabla w\equiv\mathcal{C}(u,w) \tag{6.6}$$

and

$$\mathcal{J}(u,w)\equiv -[u]\widehat{\underline{a}_n\bullet\nabla w}+\dot{\hat{w}}[\underline{a}_n\bullet\nabla u]=\mathcal{K}(u,w) \tag{6.7}$$

Observe that $P\equiv Q$, $B\equiv C$ and $J\equiv K$. The definition of the *sought information* depends on the target of information one wishes to obtain on the internal boundary and several choices are possible. Each such choice gives rise to an S^* and a corresponding formulation. For example, for the Hybrid-Trefftz finite element, the two dual formulations of Jirousek & Zielinski [33] are obtained when the sought information is chosen to be the function and the normal

derivative (or the 'traction'), respectively. For the BVPJ defined above, throughout this article the '*sought information*' will be the average of the function, on Σ. Then, the corresponding definitions of S and R are:

$$\langle Sw,u\rangle \equiv \langle S*u,w\rangle = \int_\Sigma \hat{\dot{u}}\left[a_n \nabla w\right] ds \tag{6.8}$$

and

$$\langle R*u,w\rangle \equiv \langle Rw,u\rangle = -\int_\Sigma [w]\widehat{a_n \cdot \nabla u} ds \tag{6.9}$$

Furthermore, $N \equiv N_Q \cap N_C \cap N_R \equiv N_P \cap N_B \cap N_R$ and the condition $w \in N$ is equivalent to

$$\mathcal{L}^* w = 0 \text{ in each } \Omega_j \tag{6.10}$$

$$\left[w\right] = 0, on\ \Sigma; \tag{6.11}$$

i.e., w is continuous in Ω, and

$$w = 0 \text{ on } \partial\Omega \tag{6.12}$$

It is convenient to introduce an auxiliary function $u_0 \in D$, satisfying

$$\begin{aligned} &u_0 = u = g \quad on\ \partial\Omega \\ &[u_0]_\Sigma = [u]_\Sigma = j^0 \ \ on\ \Sigma \\ &[\underline{a}_n \bullet \nabla u_0]_\Sigma = [\underline{a}_n \bullet \nabla u]_\Sigma = j^1 \ \ on\ \Sigma \end{aligned} \tag{6.13}$$

Making use of such $u_0 \in D$, we define

$$v(x) = u(x) - u_0(x), \text{ in } \Omega \tag{6.14}$$

Then

$$\mathcal{L}v(x) = f_\Omega - \mathcal{L}u_0(x), \quad \text{for } x \in \Omega_i, \quad i = 1,...,E \tag{6.15}$$

$$\begin{aligned} &v(x) = 0, \quad \text{on } \partial\Omega \\ &[v]_\Sigma = 0, \quad \text{on } \Sigma; \\ &[\underline{a}_n \bullet \nabla v]_\Sigma = 0, \text{ on } \Sigma \end{aligned} \tag{6.16}$$

In particular $Pv = f - Pu_0$, $Bv = 0$ and $Jv = 0$. Thus, Eq. (3.18) implies

$$-\langle S*v,w\rangle = \langle f,w\rangle - \langle Pu_0,w\rangle \quad \forall w \in N \tag{6.17}$$

i.e., making use of Eq.(6.8),

$$-\langle S*v,w\rangle \equiv -\int_\Sigma v\left[\underline{a}_n \bullet \nabla w\right] ds = \sum_{j=1}^{E} \int_{\Omega_j} w\left(f_\Omega - \mathcal{L}u_0\right) dx \quad \forall w \in N \tag{6.18}$$

In Eq. (6.18), the value of the function v on Σ has been used instead of the average $\dot{v}$, because $\dot{v} = v$, as v is continuous across Σ. Even more, when $f_\Omega \in H^r(\Omega)$ and $\mathcal{L}u_0(x) \in H^r(\Omega)$, with $r \geq 0$, and the coefficients of the operator $\mathcal{L}$ are sufficiently regular,

$C^2(\Omega)$ say, then $v \in H^2(\Omega)$. In particular, the trace of v on Σ belongs to $C^1(\Sigma)$. This fact will be used later on, to simplify the numerical implementation of the method. Indeed, using TH-method a numerical search for the trace of v on Σ is carried out and the fact just mentioned permits reducing the space to be searched, which is numerically advantageous. In addition, a straightforward calculation shows that when $w, \hat{w} \in N = N_Q \cap N_R \cap N_C = N_P \cap N_R \cap N_B$, then

$$-\langle S^* \hat{w}, w \rangle = -\int_\Sigma \hat{w}[\mathbf{a}_n \cdot \nabla w] dx = \sum_{i=1}^{E} \int_{\Omega_i} \left(\nabla \hat{w} \cdot \underline{\underline{a}} \cdot \nabla w + c \hat{w} w \right) dx, \ \forall \hat{w}, w \in N \tag{6.19}$$

Therefore, $-S^*$ is symmetric and positive-definite on N, whenever $c \geq 0$.

In what follows, the weak formulation of Eq. (6.18) constitutes the basic result from which TH-Domain Decomposition will be derived. When applying it, it suffices to take a TH-complete subset $\mathcal{E} \subset N$ instead of the whole of N (see [20]). The solution of the weak formulation of Eq. (3.18) yields the function v on the internal boundary Σ, exclusively. This permits deriving the average of the sought solution on the internal boundary by means of Eq. (6.14), which implies

$$\dot{u} = \dot{u}_0 + v, \text{ on } \Sigma \tag{6.20}$$

If desired, the solution u of the BVPJ can be obtained in the interior of the subdomains of the partition by 'optimal interpolation', which consists in solving well-posed Dirichlet problems in each of the subdomains of the partition. At a given subdomain, Ω_i, the boundary data consist, at points belonging to $\partial\Omega \cap \partial\Omega_i$, of the boundary data of the global problem and, at points on $\Sigma \cap \partial\Omega_i$, they are derived by application of the identities:

$$u_+ \equiv \dot{u} + \frac{1}{2}[u_\Sigma] \text{ and } u_- \equiv \dot{u} - \frac{1}{2}[u_\Sigma] \tag{6.21}$$

Observe that

$$-\langle S^* \hat{u}, \hat{w} \rangle \equiv \sum_{i=1}^{E} \int_{\partial\Omega_i} \hat{u} a_n \cdot \nabla \hat{w} dx \tag{6.22}$$

when $\hat{u} \in N_C \cap N_R$ and $\hat{w} \in N_C \cap N_R$, because

$$\begin{aligned} &\sum_{i=1}^{E} \int_{\partial\Omega_i} \hat{u} a_n \cdot \nabla \hat{w} dx = \int_{\partial\Omega} \hat{u} \frac{\partial \hat{w}}{\partial n} dx - \int_\Sigma \hat{u} [a_n \cdot \nabla \hat{w}] dx - \int_\Sigma [\hat{u}] \overline{\dot{a}_n \cdot \nabla \hat{w}} dx \\ &= \langle C\hat{u}, \hat{w} \rangle - \langle S^* \hat{u}, \hat{w} \rangle + \langle R\hat{u}, \hat{w} \rangle \end{aligned} \tag{6.23}$$

and

$$-\langle S^* \hat{u}, \hat{w} \rangle = \langle C\hat{u}, \hat{w} \rangle - \langle S^* \hat{u}, \hat{w} \rangle + \langle R\hat{u}, \hat{w} \rangle \tag{6.24}$$

Eq.(6.22) yields some advantages when numerically computing $-\langle S*\hat{u},\hat{w}\rangle$. Notice that in particular, Eq.(6.22) holds when $\hat{u}\in N$ and $\hat{w}\in N$.

7. TH-Discretization

TH-complete systems are infinite for problems in more that one independent variable. Therefore, in numerical applications of TH-Domain Decomposition, it is necessary to approximate TH-complete systems by finite families of test functions belonging to $N\equiv N_Q\cap N_C\cap N_R$. This of course implies a truncation error that is reflected in the accuracy of the approximate solutions.

Functions $w\in N$ are uniquely determined by their traces on the internal boundary Σ because they fulfill Eqs. (6.10) to (6.12). Thus, the subspace of test functions $\tilde{N}\subset N$ to be applied can be specified by taking a suitable finite dimensional manifold of dimension m, of functions defined on Σ. We use the notation $\tilde{N}\subset N$ for the subspace spanned by $\boldsymbol{\mathcal{E}}\equiv\{\tilde{w}^1,...,\tilde{w}^m\}\subset N$. Also, the trace on Σ of the function $v\in D$ that fulfills Eqs. (6.14) and (6.15) will be approximated by

$$\tilde{v}=\sum_{\alpha=1}^{m}c_\alpha\tilde{w}^\alpha\text{ , on }\Sigma \tag{7.1}$$

Here, the coefficients $\underline{c}\equiv(c_1,...,c_m)$ are determined by imposing the condition that Eq. (6.17) is fulfilled for every $\tilde{w}\in\tilde{N}$. Then

$$\underline{\underline{S}}\bullet\underline{c}=\underline{b} \tag{7.2}$$

where $\underline{b}\equiv(b_1,...,b_m)$, and

$$b_\alpha\equiv\sum_{i=1}^{E}\int_{\Omega_i}\tilde{w}^\alpha\left(f_\Omega-\mathcal{L}u_0\right)dx\text{ , }\alpha=1,...,m \tag{7.3}$$

In addition, the elements of the $(mXm)-matrix$, $\underline{\underline{S}}$, are given by

$$S_{\alpha\beta}\equiv\int_\Sigma\tilde{w}^\beta\left[\underline{a}_n\bullet\nabla\tilde{w}^\alpha\right]ds=\left\langle S\tilde{w}^\alpha,\tilde{w}^\beta\right\rangle \tag{7.4}$$

Clearly, the matrix $\underline{\underline{S}}$ is symmetric and positive definite, by virtue of Eq. (6.19).

The system of equations (7.2) was applied, in [20], using test functions with local support that are approximate solutions of the boundary value problem defined by Eqs.(6.10) to (6.12). Actually, they exactly satisfy the boundary and jump conditions, while the differential equation is approximated using orthogonal collocation on bi-cubic polynomials in the interior of the subdomains of the partition. Two algorithms were derived: one using linear interpolation on Σ and the other one piecewise cubic polynomials. This latter algorithm yields exactly the same approximation as the usual orthogonal spline collocation in bi-cubic polynomials, but with much better structured matrices. The order of approximation is $O(h^4)$; see [35]. In particular, they are symmetric and positive definite. Also, the number of degrees of freedom associated with each node is reduced to 3, from 4 (see [20]).

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