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HIGH ORDER THETA-SCHEMES FOR LINEAR WAVE EQUATIONS

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Explicit time schemes are widely used by practitioners to solve wave equations because they provide a cheap way for time integration. They are applied to semi-discrete equations of the form: $d^2u_h/dt^2 + A_h u_h = 0$, which are obtained after performing space discretizations with finite elements providing an easily invertible mass matrix. The most popular time scheme is the so-called leap-frog scheme (2a). The modified equation technique [2] yields higher order time approximations, as the fourth order scheme (2b).

$$\frac{u_h^{n+1} - 2u_h^n + u_h^{n-1}}{\Delta t^2} + A_h u_h^n = 0 \,(2a), \quad \frac{u_h^{n+1} - 2u_h^n + u_h^{n-1}}{\Delta t^2} + A_h u_h^n - \frac{\Delta t^2}{12} A_h^2 u_h^n = 0 \,(2b)$$

These schemes are conditionally stable: the time step Δt is bounded by the spectral radius $\rho(A_h)$ of A_h. The stability condition proves to be prohibitive in some cases of practical interest like heterogeneous media including strong contrasts of material properties or when mesh refinement is imposed by the geometry. An interesting alternative is provided by the second order θ-scheme (2) even though it is implicit and thus *a priori* more expensive.

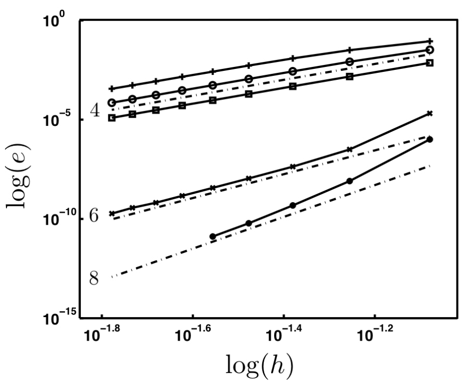
$$\frac{u_h^{n+1} - 2u_h^n + u_h^{n-1}}{\Delta t^2} + A_h \{u_h\}_\theta^n = 0, \qquad \{u_h\}_\theta^n := \theta u_h^{n+1} + (1-2\theta) u_h^n + \theta u_h^{n-1} \quad (2)$$

Energy techniques allow to prove it is unconditionally stable for $\theta \geq 1/4$. The overcost of the required matrix inversion is counterbalanced by the opportunity of using a large time step. However it has been observed that large time steps lead to a lost of accuracy. In this work, we investigate the idea of using θ as a possible parameter to control the level of accuracy. For that purpose, we modify the usual point of view which consists in choosing Δt to increase the computational accuracy for a given mesh. Herein we **pick the "best" stable numerical scheme in a given family of schemes** for a given pair $(\Delta t, \rho(A_h))$. To compare numerical schemes, we consider the coefficient of the first term of the consistency error, which is given, for the specific case of the θ-scheme (2), by $(\theta - 1/12)$. This optimization-like problem is straightforward for the usual θ-scheme: either $\Delta t^2 \rho(A_h) \leq 6$ and the value $\theta = 1/12$ leads to a fourth order stable scheme or $\Delta t^2 \rho(A_h) > 6$, and a stable scheme related to a minimized consistency error corresponds to: $\theta = 1/4 - 1/(\Delta t^2 \rho(A_h))$.

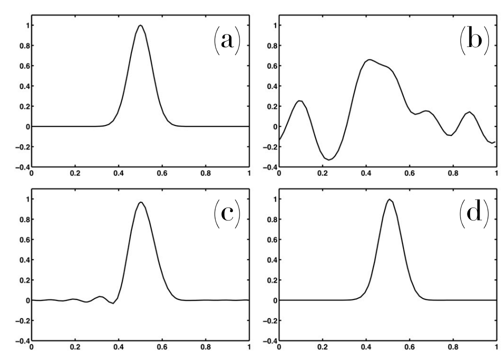
We use the modified equation technique on the usual θ-scheme to construct a new family of schemes of the form (see [1]):

$$\frac{u_h^{n+1} - 2u_h^n + u_h^{n-1}}{\Delta t^2} + A_h \left\{u_h\right\}_\theta^n + \left(\theta - \frac{1}{12}\right) \Delta t^2 A_h^2 \left\{u_h\right\}_\varphi^n = 0 \tag{3}$$

Our analysis based on the previous reasoning yields an optimal choice of the (θ, φ) coefficients which depends on the value of $\Delta t^2 \rho(A_h)$, and sometimes a higher order of accuracy is achieved (6th and 8th orders, see figure (a)). We provide numerical solution techniques and show numerical illustrations where our scheme outperforms the usual θ-scheme and the explicit scheme, see figure (b) (better accuracy for a lower computational cost).



(a) Log of the error w.r.t the log of the mesh size for optimal (θ, φ)-schemes. + "multiple roots" $\Delta t^2 \rho = 120$, $\circ$ $\Delta t^2 \rho = 120$, $\square$ $\Delta t^2 \rho = 60$, $\times$ $\Delta t^2 \rho = 26$, $\bullet$ $\Delta t^2 \rho = 22$.



(b) Snapshots of the numerical solutions at final time. (a) explicit scheme, (b) θ-scheme, (c) naive (θ, φ)-scheme, (d) optimal (θ, φ)-scheme. All implicit schemes use the time step $\Delta t = 1.6 \times 10^{-2}$.

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