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A FAMILY OF ARBITRARY ORDER MIXED METHODS FOR HETEROGENEOUS ANISOTROPIC DIFFUSION ON GENERAL MESHES

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Over the last few years, several discretization methods for diffusive problems have been proposed that support general meshes including polygonal or polyhderal elements and nonmatching interfaces. In most of the cases, such methods are obtained by preserving (or mimicking) to some extent the structure of the continuum operator at the discrete level. Instances of structure-preserving lowest-order methods are Mimetic Finite Differences [8], the Mixed/Hybrid Finite Volume method [7], and Compatible Discrete Operators [3]. The similarities among the above-mentioned approaches and with other methods have been highlighted in several papers; cf., e.g., [3,6,7]. A rather different point of view from the previous works is adopted in [5], where the application of interior penalty strategies to consistent reconstructions of differential operators is considered.

Until recently, the main focus has been on lowest-order methods. Very recent works consider, however, the extension to higher orders. An example is provided by the arbitrary-order nodal mimetic method of [2]. We also recall here the Virtual Element method, whose basic principles are exposed in [1]; cf. also references therein. The adjective "virtual" refers here to the fact that one defines a variational formulation in a finite element fashion, but without explicitly defining the underlying basis functions. We also mention here a recent work on mimetic products of discrete differential forms [4], which also contains an extensive bibliographic section.

We propose here a new family of arbitrary order mixed methods for anisotropic heterogeneous diffusion on general polyhedral meshes. A key point in the definition of the methods is the choice of flux degrees of freedom, which, for a given integer $k \geq 0$, are selected to be the fluxes of polynomial potentials of degree $\leq k$ inside cells and polynomials of degree $\leq k$ at faces. Cell degrees of freedom are hence not used in the lowest-order case $k = 0$. Based on these degrees of freedom we reconstruct (i) a discrete divergence D_h^k which satisfies the usual commuting diagram property for potentials that are broken polynomials of degree $\leq k$ and (ii) a flux reconstruction that is exact when the potential is a polynomial of degree $\leq (k+1)$ inside each element (*consistency*) and has coercivity properties on the kernel of D_h^k (*stability*). Stability is achieved by penalizing residuals inside a pyramidal submesh of each element. The flux reconstruction and the divergence are then used to define the discrete counterparts of the bilinear forms that appear in the continuous mixed formulation. Under the usual regularity assumptions for the exact solution, it is proved that the error on the flux converges as h^{k+1} (h denotes here the meshsize). Additionally, provided elliptic regularity holds, one can prove a supercloseness result for the potential as in classical mixed finite element methods, meaning that the distance from the L^2-orthogonal projection of the exact potential converges as h^{k+2} . Several variations are considered including, in particular, virtual versions which do not require to solve local problems for the stabilization term. The link with existing methods in the lowest-order case is also discussed. Figure 1 shows convergence rates for the approximate solution on the unit square $\Omega = (0,1)^2$ of the homogeneous Dirichlet problem with exact solution $u = \sin(\pi x_1)\sin(\pi x_2)$ and unit diagonal diffusion tensor on the hexagonal mesh family of [6, Section 4.2.3].

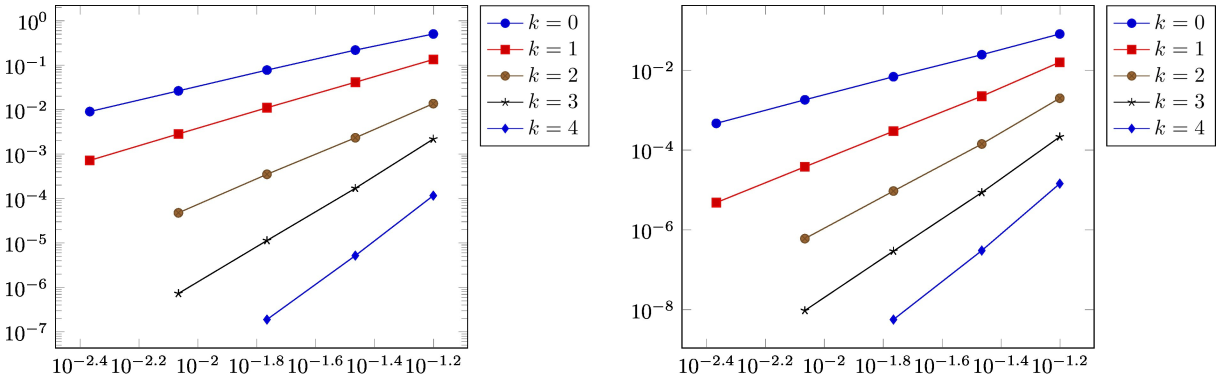

Figure 1. Convergence results

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