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APPLICATION OF SPLITTING AND FINITE-VOLUME METHODS FOR SOLUTION OF ADVECTION-DIFFUSION EQUATION ON A SPHERE

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An efficient non-iterative numerical algorithm of second order approximation in spatial and temporal variables is suggested for the solution of advection-diffusion equation on the surface of a two-dimensional sphere:

$$\phi_t + \mathrm{div}(\boldsymbol{U}\phi) + \sigma\phi - \nabla\cdot(\mu\,\nabla\phi) = f\,, \qquad \phi(x,0) = \phi^0(x)\,.$$

Here $\phi(x,t)$ is a physical substance. The new method can be applied to problems of global transport of various passive pollutants, temperature, humidity, etc. in the atmosphere [1,2].

The vector field of velocity is assumed to be known and non-divergent. The spherical regular grid is used, and the surface of sphere is partitioned into a set of non-overlapping grid cells of trapezoidal form and two round pole cells. The discretization of advection-diffusion equation in space is performed with the finite-volume method by applying Gauss's theorem to each grid cell. The discretization in time is carried out in each double-step interval by using the symmetrized (double cyclic) splitting method by Marchuk [3,4] which is based on the application of Crank-Nicolson scheme. The numerical scheme obtained is of second order approximation in space and time.

The sum of discrete equations over all grid points gives the equation of balance of mass in the discrete system. Since every one-dimensional numerical scheme is unconditionally stable, the whole numerical algorithm is also unconditionally stable. The numerical scheme correctly describes the behavior of the total mass and the discrete analogue of L_2-norm of solution in the forced and dissipative discrete system. Besides, in the particular case of closed discrete system (in the absence of dissipation and external forcing), both the total mass and the norm of solution are conserved in time. Due to Lax's theorem [5], the numerical solution converges to the solution of the original continuous problem at a rate that increases quadratically with decreasing the sizes of the spatial-temporal grid.

The matrix of each of the one-dimensional discrete problems, obtained in the process of splitting with respect to longitudinal and latitudinal directions, is positive semi-definite, and hence, any of these problems has the only solution. Besides, every problem in the longitudinal direction is periodic, and is solved by using Sherman-Morrison's formula [6]. And the problems in the latitudinal direction are solved from the North Pole to the South Pole by the bordering method that requires a prior definition of the problem solution in both poles. The application of the bordering method leads to a set of problems of linear algebra with tridiagonal matrices (along each meridian), which are solved by using Thomas's factorization method [7]. The new algorithm is direct (without iterations), efficient and rapid in realization. Parallel processors can be used in solving the systems of split equations at each splitting step.

This method can also be applied for solving pure advection problems, linear and nonlinear diffusion problems on a sphere and stationary elliptic problems on a sphere of the form

$$\sigma\phi - \nabla\cdot(\mu\nabla\phi) = f(x) .$$

In the latter case, the solution is sought as the limit as $t \to \infty$, of the solution $\psi(x,t)$ of the following non-stationary problem

$$\psi_t = \nabla\cdot(\mu\nabla\psi) - \sigma\psi = f(x)$$

with arbitrary initial condition.

REFERENCES

[1] Yu.N. Skiba and D. Parra-Guevara, *Introducción a los métodos de dispersión y control de contaminantes*. Dirección General de Publicaciones y Fomento Editorial, *UNAM*, 1ra Edición, México, 424 pp., 2011.

[2] G.I. Marchuk, *Mathematical models in environmental problems*, North-Holland, Elsevier, 217 pp., 1986.

[3] G.I. Marchuk, *Methods of Numerical Mathematics*, 2nd Edition, Springer, 1982.

[4] Yu.N. Skiba, *Métodos y Esquemas Numéricos: Un Análisis Computacional*. Dirección General de Publicaciones y Fomento Editorial, UNAM, 1ra Edición, México, 440 pp., 2005.

[5] P. Lax and R.D. Richtmyer, Servey of the stability of linear finite difference equations. *Comm. Pure Appl. Math.*, Vol. **9**, pp. 267-293, 1956.

[6] J. Sherman and W.J. Morrison, Adjustment of an Inverse Matrix Corresponding to a Change in One Element of a Given Matrix. *Ann. Math. Statistics*, Vol. **21**, pp. 124-127, 1950.

[7] L.H. Thomas, *Elliptic Problems in Linear Differential Equations over a Network*, Watson Sci. Comput. Lab Report, Columbia University, New York, 1949.