11th World Congress on Computational Mechanics (WCCM XI)
5th European Conference on Computational Mechanics (ECCM V)
6th European Conference on Computational Fluid Dynamics (ECFD VI)
July 20–25, 2014, Barcelona, Spain

GOAL-BASED ADAPTIVE COUPLING STOCHASTIC AND DETERMINISTIC ERRORS IN COMPRESSIBLE CFD

Anca C. Belme[1] **and Didier Lucor**[2]

[1] Sorbonne Universités, UPMC Univ Paris 06, UMR 7190, Institut Jean le Rond d'Alembert, F-75005, Paris, France
[2] CNRS, UMR 7190, Institut Jean le Rond d'Alembert, F-75005, Paris, France

Key words: *error control, uncertainty quantification, adjoint, mesh adaptivity, non-intrusive gPC*

A strategy for goal-oriented error estimation of the coupled deterministic and stochastic errors is proposed in this paper. Furthermore, an adaptive method is proposed to enhance the quality of a functional of interest obtained by the coupled stochastic-deterministic solution of a parametrised system. Recent works have extended goal-based methodology to problems with uncertain data, mostly by extending the well-known dual-weighted a posteriori error estimation to stochastic error. For instance O. L. Le Maître *et al* proposed in [4] such a goal-oriented error estimator using the discrete solution of the primal stochastic problem and two discrete adjoint solution (on two imbricated spaces) of the dual stochastic problem. Their error estimate has been used as an indicator for p or isotropic-h refinement in the deterministic and respectively parametric space. Altough this is a first attempt to adress the two error sources, several non-trivial questions remain open: which error dominates the simulation and how to capture the anisotropy in both deterministic and parametric spaces. S. Prudhomme *et al* adressed these questions in [3] and proposed an error estimator where the global error is split in contribution from the deterministic and stochastic approximation, and proposes several algorihms for p or h refinement of the parametric space, while an uniform h-refinement is proposed for the deterministic space. Our goal is to built a goal-oriented method for anisotropic adaptive control of deterministic and stochastic errors. The anisotropy in the deterministic space is controled for a functional of interest via a riemannian metric (hessian) based method and a similar approach is extended to stochastic errors.
We consider an abstract model problem defined on an open bounded domain $D \in \mathcal{R}^d$. We focus here on PDE's with uncertain parameters denoted by the random vector $\xi = \xi_{\mathbf{i}}(\theta)_{\mathbf{i=1}}^{\mathbf{n}}$. We introduce a probability space (Θ, Σ, P), with Θ the sample space, Σ a σ-algebra on Θ, and P the probability measure. The abstract model problem can be cast in general, residual form as,

$$R(w(\mathbf{x}, \xi)) = 0, \ \forall \mathbf{x} \in D. \tag{1}$$

Here R contains the boundary and initial conditions and $w(\mathbf{x}, \xi)$ is the solution of the continous model in the appropiate functional space $\mathcal{V} = \mathcal{V}_D \oplus \mathcal{V}_\Theta$. Out focus is on a functional of interest j, represented as a functional of the random process w: $j(w(.,\xi)) = (g, w(.,\xi))_D$. The total error commited on the target functional writes then as:

$$\begin{aligned} \delta j &= j(w(.,\xi)) - j(w_h^N(.,\xi)) = (g, w(.,\xi) - w_h^N(.,\xi)) \\ &= \underbrace{(g, w(.,\xi) - w_h(.,\xi))}_{\text{spatial error}} + \underbrace{(g, w_h(.,\xi) - w_h^N(.,\xi))}_{\text{implicit stochastic error}} \end{aligned} \tag{2}$$

where $w_h^N(\mathbf{x}, \xi)$ is the fully discretised solution obtained from a finite-element (or volume) discretisation of computational domain and generalised polynomial chaos (gPC) projection in the parametric space. We

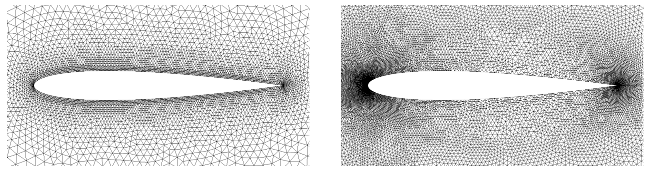

Table 1: NACA0012: uniform initial mesh (left image) and goal-oriented adapted mesh (right image)

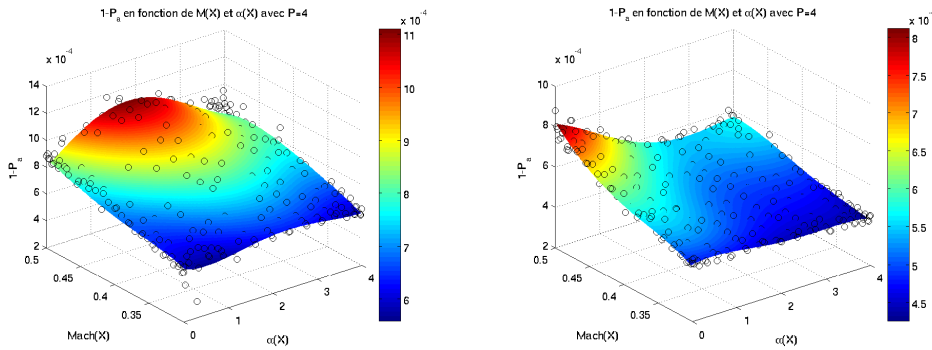

Table 2: Functional error for adaptive mesh of about 10000 nodes (left) and respectively 20000 (right)

remark that we can split the total error in contribution from spatial error and contribution from stochastic discretisation of the already spatially discretised solution. Our purpose is to control both spatial and stochastic error in an automatic, adaptive way. The spatial error δj^h will be controled through mesh adaptivity deriving an "optimal" mesh from an optimal riemannian metric, a matrix tensor defined for each node as a hessian weighted by the adjoint state associated to the dual problem. We will use the *a priori* error estimator in [1] and show that :

$$\delta j^h \approx (w^*(.,\xi),(R_h - R)(w(.,\xi))) \tag{3}$$

where $w^*(.,\xi)$ holds for the adjoint state, a random process solution of the dual parametrised problem. In a similar manner we can derive a goal-oriented error estimator for the *implicit stochastic error*:

$$\delta j^N \approx (w_h^{*,N},(R_h^N - R_h^{N+k})(w_h^{N+k})) \tag{4}$$

where R_h^N and R_h^{N+k} are respectively residual associated to the approximate solution w_h^N and a higher (optimal) projection order solution w_h^{N+k}. The expansion (truncation) order of the gPC projection is increased in case a target error criterion is not satisfied.
Performance of the proposed approach is illustrated on 2D and 3D numerical experiments. An overview of the results obtained for the inviscid compressible flow simulation around a 2D NACA0012 airfoil is presented in the images bellow. The Mach number and the angle of attack α are the uncertain parameters considered for this study and the functional of interest is the stagnation pressure. The initial and respectively anisotropic adapted mesh for are picturised in Figure 1. 17 quadrature points are used for Mach $\in [0.3, 0.5]$ and $\alpha \in [0., 4.]$ and an adaptive spatial mesh is derived for each calculation. The improvement of mesh adaptivity on the uncertainties quantification is shown in Figure 2 where we show the error on our functional decreases considerably compared to the uniform case which gives a more accurate stochastic response.

REFERENCES

[1] Anca C. Belme, Frédéric Alauzet and Alain Dervieux. Time accurate anisotropic goal-oriented mesh adaptation for unsteady flows. *Journal of Comp. Physics*, Vol. 231-19, 6323–6348, 2012

[2] T. Butler, P. Constantine and T. Wildey. A posteriori error analysis of parametrized linear systems using spectral methods.*SIAM J. MATRIX ANAL. APPL.*, Vol. 33-1, 195-209, 2012

[3] Corey M. Bryant, Serge Prudhomme and Timothy Wildey. A posteriori error control for partial differential equations with random data. *ICES Report.*, 13-08, The Institute for Computational Engineering and Sciences, The University of Texas at Austin, April 2013

[4] L. Mathelin and O. L. Le Maître. Dual-based a posteriori error estimate for stochastic finite element methods. *COMM. APP. MATH. AND COMP. SCI.*, Vol. **2**, No. 1, 83–115, 2007