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RATE OPTIMALITY OF ADAPTIVE ALGORITHMS: AN AXIOMATIC APPROACH

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Adaptive mesh-refining algorithms dominate the numerical simulations in computational sciences and engineering, because they promise optimal convergence rates in an overwhelming numerical evidence. The mathematical foundation of optimal convergence rates has recently been completed [3] and shall be discussed in this presentation. We aim at a simultaneous axiomatic presentation of the proof of optimal convergence rates for adaptive finite elements as well as boundary elements in the spirit of [1]. For this purpose, an overall set of four axioms on the error estimator is sufficient and (partially even) necessary.

The abstract setting concerns an estimator $\eta(\mathcal{T})$ for each admissible triangulation, written $\mathcal{T} \in \mathbb{T}$. This estimator allows a piecewise error estimation in the sense that

$$\eta(\mathcal{T})^2 = \sum_{T \in \mathcal{T}} \eta(\mathcal{T}; T)^2$$

and hence the Dörfler marking which selects a set $\mathcal{M} \subseteq \mathcal{T}$ on the current level ℓ of the successive adaptive mesh-refining such that

$$\theta\, \eta(\mathcal{T})^2 \le \sum_{T \in \mathcal{M}} \eta(\mathcal{T}; T)^2$$

for some sufficiently small and positive bulk parameter $0 < \theta < 1$. Given the adaptive algorithm, the computed sequence of triangulations $\mathcal{T}_\ell$ and the corresponding estimators $\eta_\ell := \eta(\mathcal{T}_\ell)$ are compared with the optimal choices amongst the triangulations

$$\mathbb{T}(N) := \{\mathcal{T} \in \mathbb{T} : |\mathcal{T}| \le |\mathcal{T}_0| + N\}$$

with at most N (refined) element domains more than $\mathcal{T}_0$; here, $\mathcal{T}_0$ is the initial triangulation which generates the set $\mathbb{T}$ of admissible triangulations by newest vertex bisection, and $|\bullet|$ denotes the number of simplices $|\mathcal{T}|$ in the triangulation $\mathcal{T}$.

The presentation discusses four axioms which are formulated for the estimators $\eta := (\eta(\mathcal{T}))_{\mathcal{T}\in\mathbb{T}}$ only and which guarantee the optimal rates in the sense that, for any rate $0 < \sigma < \infty$,

$$|\eta|_\sigma := \sup_{N\in\mathbb{N}_0} (1+N)^\sigma \min_{\mathcal{T}\in\mathbb{T}(N)} \eta(\mathcal{T}) < \infty$$

implies that the computed values of the estimator converge with the same rate in the sense that

$$\sup_{\ell\in\mathbb{N}_0} (1 + |\mathcal{T}_\ell| - |\mathcal{T}_0|))^\sigma \eta_\ell \le C\,|\eta|_\sigma$$

for some universal constant $C = C(\mathcal{T}_0, \sigma, \theta)$ and sufficiently small θ.

The four axioms are stability on non-refined element domains (A1), reduction on refined element domains (A2), general quasi-orthogonality (A3), and discrete reliability (A4). The presentation shall discuss those properties in very simple examples and motivate the different arguments which lead to optimal convergence rates. It can be shown that the convergence rates of the estimators are optimal in terms of nonlinear approximation classes. In case of efficiency, one discovers the former results from the literature, e.g., [2].

Compared to the state of the art in the temporary literature [1, 2], the improvements of [3] can be summarized as follows: First, a general framework is presented which covers the existing literature on rate optimality of adaptive schemes for both, linear as well as nonlinear problems, which is fairly independent of the underlying finite element or boundary element method. Second, efficiency of the error estimator is not needed. Instead, efficiency exclusively characterizes the approximation classes involved in terms of the bestapproximation error plus data resolution. Third, some general quasi-Galerkin orthogonality is not only sufficient, but also necessary for the R-linear convergence of the error estimator, which is a fundamental ingredient in the current quasi-optimality analysis [1, 2]. Finally, the general analysis allows for various generalizations like equivalent error estimators and inexact solvers as well as different non-homogeneous and mixed boundary conditions.

REFERENCES

[1] Stevenson, R., *Optimality of a standard adaptive finite element method.* Found. Comput. Math., vol. 7, pp. 245–269, (2007).

[2] Cascon, J.M., Kreuzer, C., Nochetto, R.H., and Siebert, K., *Quasi-optimal convergence rate for an adaptive finite element method.* SIAM J. Numer. Anal., vol. 46, pp. 2524–2550, (2008).

[3] Carstensen, C., Feischl, M., Page, M., and Praetorius, D., *Axioms of adaptivity.* Comput. Math. Appl., vol. 67, pp. 1195–1253, (2014).
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