

GEOMETRICALLY EXACT FINITE ELEMENTS FOR REISSNER BEAMS: THE EXACT DYNAMICS FOR HELICAL INTERPOLATION

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In this abstract we will present the main strategies we followed in order to obtain analytically the exact expression of the mass matrix and its derivatives for the dynamics of geometrically exact (GE) Reissner beams with the helical interpolation.

This GE beam model was firstly developed by Simo [?] and its main idea is to exploit the geometric properties of the spaces appearing in the continuous formulation of the problem. Following Reissner beam kinematics the configuration of the beam can be expressed through a function

$$\begin{aligned}
 H &: I \subset \mathbb{R} \rightarrow SE(3) \\
 u &\mapsto H(u) = \begin{pmatrix} R(u) & \mathbf{p}(u) \\ \mathbf{0}^t & 1 \end{pmatrix}, \quad (1)
 \end{aligned}$$

where $\mathbf{p} \in \mathbb{R}^3$ describes the position of the line of centroids and $R(u) \in SO(3)$ the rotation of the section at $\mathbf{p}(u)$. To define a displacement-based FEM, the interpolation chosen here is the helical one (already presented in [?] but not with the formalism of $SE(3)$), i.e.

$$H(u) = H_i \exp_{SE(3)} \left(\frac{u}{L_{ij}} \log_{SE(3)} \left(H_i^j \right) \right) \quad \forall u \in [0, L_{ij}], \quad (2)$$

where $H_i^j = H_i^{-1} H_j \in SE(3)$, L_{ij} is the length of the element ij , $u \in [0, L_{ij}]$ is the material coordinate on the element and $H_k \in SE(3)$ is the position of the nodes k , $k = i, j$. It is worth noting that this interpolator follows the main idea of GE beam models, is left-invariant and, in contrast with the interpolator used by Simo in [?], it is objective. As pointed out in [?], in displacement-based FE for beams, the mass matrix exhibits nonlinearities and is not constant with respect to nodal positions, except in the case of the absolute nodal coordinate method.

Assuming that the cross section is constant along the beam, the kinetic energy is

$$E_C = \frac{1}{2} \int_0^{L_{ij}} \mathbf{T}(u)^t \begin{pmatrix} X & 0 \\ 0 & \mu I_3 \end{pmatrix} \mathbf{T}(u) du, \quad (3)$$

where $X \in \mathbb{R}^{3 \times 3}$ is the inertia matrix of the cross section, $I_3 \in \mathbb{R}^{3 \times 3}$ the identity matrix, μ the linear mass density and $\mathbf{T} = (\boldsymbol{\omega}, \mathbf{v}) \in \mathbb{R}^6$ the vector of material velocities defined as

$$d_t H(u) := H^{-1}(u) \frac{dH(u)}{dt} = \widehat{\mathbf{T}}(u) = \begin{pmatrix} \widehat{\boldsymbol{\omega}} & \mathbf{v} \\ 0 & 0 \end{pmatrix} \in \mathfrak{se}(3), \quad \widehat{\boldsymbol{\omega}} = \begin{pmatrix} 0 & -\omega_3 & \omega_1 \\ \omega_3 & 0 & -\omega_2 \\ -\omega_1 & \omega_2 & 0 \end{pmatrix}. \quad (4)$$

Using the choosen interpolation, $\widehat{\mathbf{T}}$ has the following expression

$$\widehat{\mathbf{T}} = Ad_{\exp\left(-\frac{u}{L_{ij}} \log H_i^j\right)} \widehat{\mathbf{T}}_i^{i,0} + \frac{u}{L_{ij}} d \exp\left(\frac{u}{L_{ij}} \log H_i^j\right) d \log H_i^j \left(\widehat{\mathbf{T}}_j^{j,0} - Ad_{H_i^j} \widehat{\mathbf{T}}_i^{i,0} \right). \quad (5)$$

In order to find the mass matrix, we have to rewrite the kinetic energy as a quadratic form on the material velocities at the nodes $\widehat{\mathbf{T}}_k^{k,0} = H_k^{-1} \dot{H}_k$, $k = i, j$, i.e.

$$E_C = \frac{1}{2} \begin{pmatrix} \mathbf{T}_i^{i,0} \\ \mathbf{T}_j^{j,0} \end{pmatrix}^t M(H_i^j, L_{ij}) \begin{pmatrix} \mathbf{T}_i^{i,0} \\ \mathbf{T}_j^{j,0} \end{pmatrix}, \quad M \in \mathbb{R}^{12 \times 12}. \quad (6)$$

If we try to arrive at this form directly with a brute force approach, we quickly find heavy and long expressions that rapidly become unmanageable even if the exponential map on $SE(3)$ and its derivatives have closed forms derived from Rodrigues' formula.

Based on a deep insight into the different terms we have to deal with, our strategy consists in defining two auxiliary functions:

$$\alpha_1(\lambda) = \frac{1 - \cos \lambda}{\lambda \|\boldsymbol{\omega}_i^j\|} \quad \text{and} \quad \alpha_2(\lambda) = \frac{\lambda - \sin \lambda}{\|\boldsymbol{\omega}_i^j\|^2 \lambda}, \quad (7)$$

where $\widehat{\Omega}_i^j = \log(H_i^{-1} H_j)$, $\Omega_i^j = (\boldsymbol{\omega}_i^j, \mathbf{v}_i^j) \in \mathbb{R}^6$ and $\lambda = \|\frac{u}{L_{ij}} \boldsymbol{\omega}_i^j\|$. Thanks to these functions, we can write every scalar term appearing in the integrals in E_C as a polynomial in α_1 and α_2 . It turns out that we can eliminate all the terms like α_2^k with $k \geq 2$, reducing the number of integrals we have to calculate. All these integrals can be written as polynomials in $\beta_1 = \alpha_1(\|\boldsymbol{\omega}_i^j\|)$ and $\beta_2 = \alpha_2(\|\boldsymbol{\omega}_i^j\|)$. This strategy proves to be really efficient to calculate and implement not only the mass matrix, but also its derivatives with respect to nodal positions. Ongoing and future works include benchmarking of this FE and comparison with other variants of this GE interpolator, especially in $SO(3) \times \mathbb{R}^3$.

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