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HIGH-ORDER INTEGRATION FOR FLEXURAL WAVE EQUATION AND DISPERSION

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The objective of this paper is to present the wave velocity dispersion produced by an efficient high-order step-by-step direct integration algorithm for flexural wave equation. The proposed algorithm is formulated in terms of two Hermitian finite difference operators of fifth-order local truncation error and it is unconditionally stable with no numerical damping presenting a fourth-order truncation error for period dispersion (global error). Although it is in competition with higher-order algorithms presented in the literature, the computational effort is similar to that of the classical second-order Newmark's method [1].

The finite element method is used to solve a wide range of engineering problems including wave propagation in elastic media. As the finite element method is an approximation of a continuum medium, its employment to solve wave motion results in appendant dispersion [2]. It is known that numerical wave velocity dispersion normally occur in finite element solutions, when the wave equations are first semi-discretized in space by using Galerkin methodology and then numerically integrated in time.

The classical Timoshenko´s flexural wave equations can be written as [3]

$$\begin{aligned} kGS\left(v^{II}-\alpha^{I}\right)-m\ddot{v}=0 \\ -EI\alpha^{II}-kGS(v^{I}-\alpha)+mr^{2}\ddot{\alpha}=0 \end{aligned} \tag{1}$$

where v and α represent deflection and bending rotations, respectively; k, S, E, I, m and r represent the shear coefficient, the beam section area, the shear elasticity modulus, the second moment of area of the beam section and radius of gyration, respectively. The classical space derivative roman number notation as an exponent is employed [4], as well the time derivative by single dot.

The integration in non-finite terms (wave solution) of equation (1) in terms of complex notation is given by

$$\begin{aligned} v=A\exp\left[i(\beta x-\omega t)\right] \\ \alpha=B\exp\left[i(\beta x-\omega t)\right] \end{aligned} \tag{2}$$

where i is the complex unit, λ is the wave length, c is the wave speed, A and B are the deflection and bending rotation amplitudes of the wave which propagate in the positive direction, respectively.

The step-by-step integration algorithm considered in this paper is derived by considering the following coupled Hermitian operator [3]:

$$\begin{aligned} 12\left(u_k - u_{k+1}\right) + 6\Delta t\left(\dot{u}_k + \dot{u}_{k+1}\right) + \Delta t^2\left(\ddot{u}_k - \ddot{u}_{k+1}\right) = 0 \\ 12\left(\dot{u}_k - \dot{u}_{k+1}\right) + 6\Delta t\left(\ddot{u}_k + \ddot{u}_{k+1}\right) + \Delta t^2\left(\dddot{u}_k - \dddot{u}_{k+1}\right) = 0 \end{aligned} \tag{3}$$

where u indicates the function to be integrated and Δt is the time step. Integration in space of equation (1) by two node beam finite element method and integration in time using equation (3) after some algebraic manipulations for a given frequency gives the following typical fourth degree algebraic equation (eigen-value problem):

$$A_4 \cos^4\left(\beta_n \ell\right) + A_3 \cos^3\left(\beta_n \ell\right) + A_2 \cos^2\left(\beta_n \ell\right) + A_1 \cos\left(\beta_n \ell\right) + A_0 = 0 \tag{4}$$

where β_n is the numerical wave number, A_i are parameters which depend on trigonometric functions of $\omega\Delta t$ and the following mesh parameters

$$\begin{aligned} \Psi_1 &= \left(\frac{r\delta}{\ell a}\right)^2 \frac{1}{\Phi} \\ \Psi_2 &= \left(\frac{\delta}{a}\right)^2 \frac{1}{\Phi} \\ \Phi &= \frac{12E}{kG}\left(\frac{r}{\ell}\right)^2 \end{aligned} \tag{5}$$

in which ℓ is the element length, r is the radius of gyration of the section of the beam, E is the modulus of elasticity, G is the shear elasticity modulus $a = T/\Delta t$, T is the wave period, $\delta = \lambda/\ell$ and λ is the wave length.

To illustrate the numerical wave number dispersion let us consider a WF shape with the properties $E = 2.1x10^{11}\,N/m^2$, $G = E/2.6$, $I = 3.89x10^{-5}\,m^4$, $r = 0.15m$ and $\rho = 7850Kg/m^3$ The results shown in Table 1 indicate that for frequency 10KHz the proposed method presents accuracy in competition with classical Newmark method.

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Table 1 Numerical wave number

	a=b	exact	Proposed	Newmark
First spectrum wave	100	27.8899	27.9140	27.8752
	50		27.9858	27.8313
	10		31.0968	26.7243

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